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Hypercomplex Models of Multichannel Images

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Abstract—We present a new theoretical approach to the processing of multidimensional and multicomponent images based on the theory of commutative hypercomplex algebras, which generalize the algebra of complex numbers. The main goal of the paper is to show that commutative hypercomplex numbers can be used in multichannel image processing in a natural and effective manner. We suppose that animal brains operate with hypercomplex numbers when processing multichannel retinal images. In our approach, each multichannel pixel is regarded as a K-dimensional (KD) hypercomplex number rather than a KD vector, where K is the number of different optical channels. This creates an effective mathematical basis for various function–number transformations of multichannel images and invariant pattern recognition.

Keywords: multichannel images, hypercomplex algebras, image processing.

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Dedicated to the mathematician, philosopher, and poet Vladimir Danilovich Mazurov on his eightieth birthday.

INTRODUCTION

Multichannel images are widely applied in Earth remote sensing systems for solving various scientific and applied problems (see [1–4]). In this paper, we propose novel models for multichannel images using commutative hypercomplex algebras. The term "multichannel images" is used to denote images with more than one component. They are composed of a series of images $f_{\lambda_0}(\mathbf{x}), f_{\lambda_1}(\mathbf{x}), \ldots, f_{\lambda_{K-1}}(\mathbf{x})$ obtained in different optical ranges at wavelengths $\lambda_0, \lambda_1, \ldots, \lambda_{K-1}$, which are called spectral channels, where K is the number of different optical channels. A simple example is a color image $\mathbf{f}_{\mathrm{Col}}(x,y) = (f_R(x,y), f_G(x,y), f_B(x,y))$ with red $f_R(x,y)$, green $f_G(x,y)$, and blue $f_B(x,y)$ components. If an image is assembled from a small number of channels (less than ten), then it is called *multispectral*, whereas an image consisting of several tens or hundreds of channels is called *hyperspectral* (of course, this classification is conventional). Multichannel images are considered as n-dimensional (nD) K-component (vector-valued) signals

$$\mathbf{f}(\mathbf{x}) = (f_0(\mathbf{x}), f_1(\mathbf{x}), \ldots, f_{K-1}(\mathbf{x}))\colon \mathbf{R}^n \to \mathbf{V}^K$$

with values lying in a KD perceptual vector space $\mathbf{V}^K$, where $\mathbf{x} \in \mathbf{R}^n$ for $n = 2, 3, \ldots$. The following cases are the most interesting:

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(1) 2D and 3D bichromatic images

$$\mathbf{f}(\mathbf{x}) = (f_0(x_1,x_2), f_1(x_1,x_2))\colon \mathbf{R}^2 \to \mathbf{V}^2, \quad \mathbf{f}(\mathbf{x}) = (f_0(x_1,x_2,x_3), f_1(x_1,x_2,x_3))\colon \mathbf{R}^3 \to \mathbf{V}^2;$$

(2) 2D and 3D trichromatic (color) images

$$\mathbf{f}(\mathbf{x}) = (f_0(x_1,x_2), f_1(x_1,x_2), f_2(x_1,x_2))\colon \mathbf{R}^2 \to \mathbf{V}^3_{\mathrm{rgb}},$$

$$\mathbf{f}(\mathbf{x}) = (f_0(x_1,x_2,x_3), f_1(x_1,x_2,x_3), f_2(x_1,x_2,x_3))\colon \mathbf{R}^3 \to \mathbf{V}^3_{\mathrm{rgb}};$$

(3) 2D and 3D K-channel images

$$\mathbf{f}(\mathbf{x}) = (f_0(x_1,x_2), f_1(x_1,x_2), f_2(x_1,x_2))\colon \mathbf{R}^2 \to \mathbf{V}^3_{\mathrm{rgb}},$$

$$\mathbf{f}(\mathbf{x}) = (f_0(x_1,x_2,x_3), f_1(x_1,x_2,x_3), f_2(x_1,x_2,x_3))\colon \mathbf{R}^3 \to \mathbf{V}^3_{\mathrm{rgb}};$$

(4) 2D and 3D bichromatic binocular (two-camera) images (see [5, 6])

$$\mathbf{f}^{\mathrm{Bin}}(x_1,x_2) = (\mathbf{f}^L(\mathbf{x}), \mathbf{f}^R(\mathbf{x}))\colon \mathbf{R}^2 \to {}^L\mathbf{V}^2 \oplus {}^R\mathbf{V}^2,$$

$$\mathbf{f}^{\mathrm{Bin}}(x_1,x_2,x_3) = (\mathbf{f}^L(\mathbf{x}), \mathbf{f}^R(\mathbf{x}))\colon \mathbf{R}^3 \to {}^L\mathbf{V}^2 \oplus {}^R\mathbf{V}^2;$$

(5) 2D and 3D trichromatic (color) binocular images

$$\mathbf{f}^{\mathrm{Bin}}(x_1,x_2) = (\mathbf{f}^L(\mathbf{x}), \mathbf{f}^R(\mathbf{x}))\colon \mathbf{R}^2 \to {}^L\mathbf{V}^3_{\mathrm{rgb}} \oplus {}^R\mathbf{V}^3_{\mathrm{rgb}},$$

$$\mathbf{f}^{\mathrm{Bin}}(x_1,x_2,x_3) = (\mathbf{f}^L(\mathbf{x}), \mathbf{f}^R(\mathbf{x}))\colon \mathbf{R}^3 \to {}^L\mathbf{V}^3_{\mathrm{rgb}} \oplus {}^R\mathbf{V}^3_{\mathrm{rgb}};$$

(6) 2D and 3D K-channel binocular images (see [5, 6])

$$\mathbf{f}^{\mathrm{Bin}}(x_1,x_2) = (\mathbf{f}^L(\mathbf{x}), \mathbf{f}^R(\mathbf{x}))\colon \mathbf{R}^2 \to {}^L\mathbf{V}^K \oplus {}^R\mathbf{V}^K,$$

$$\mathbf{f}^{\mathrm{Bin}}(x_1,x_2,x_3) = (\mathbf{f}^L(\mathbf{x}), \mathbf{f}^R(\mathbf{x}))\colon \mathbf{R}^3 \to {}^L\mathbf{V}^K \oplus {}^R\mathbf{V}^K,$$

where $\mathbf{f}^L(\mathbf{x})$ and $\mathbf{f}^R(\mathbf{x})$ are the images emerging on the retina of the left and right eye, respectively.

For image processing and recognition, we turn the perceptual spaces $\mathbf{V}^K$ into the corresponding hypercomplex algebras (and call them *perceptual algebras*). We develop algebraic models for two brain levels (the first level is the retina and the second is the visual cortex (VC)) using different hypercomplex algebras: commutative algebras for the first level, where image processing and transformation occur, and noncommutative algebras for the second level, where the images are recognized. The use of noncommutative algebras is related to the fact that many geometric transformations of images (simultaneous dilatations, rotations, and affine and projective transformations) belong to noncommutative groups. It turns out that each such transformation can be described by an appropriate multidimensional hypercomplex number (for example, a quaternion for rotations of 3D images).

One of our hypotheses is that the brain of animals must have innate knowledge of such numbers and be able to operate with them in the pattern recognition mode. In the next part of this study, we will show that algebraic models of multichannel images allow us to develop simple, intuitive, and efficient (on the computational side) invariant algorithms for recognition of such images using the fast Fourier–Clifford–Galois transforms.

In the proposed algebraic–geometric approach, each multichannel pixel is considered as a KD hypercomplex number rather than a KD vector (note that the numerical nature does not negate the vector nature of a hypercomplex number: the vector space is simply equipped with the operation of vector multiplication of vectors, which are interpreted as numbers).

In this paper, we interpret a multichannel retinal image as a multiplet-valued signal

$$\mathbf{f}(\mathbf{x}) = (f_0(\mathbf{x}), f_1(\mathbf{x}), \ldots, f_{K-1}(\mathbf{x})) = \sum_{s=0}^{K-1} f_s(\mathbf{x})\varepsilon^s = f_0(\mathbf{x})\varepsilon^0 + f_1(\mathbf{x})\varepsilon^1 + \ldots + f_{K-1}(\mathbf{x})\varepsilon^{K-1},$$

which takes value in one of the three commutative algebras

$$\mathrm{Alg}_K^{\mathrm{Ret}}(\mathbf{R}\,|\,\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^{K-1}) = \mathrm{Alg}_K^{\mathrm{Ret}},$$

where $\varepsilon^K = -1,\ 0,$ or $+1$. Here $\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^{K-1}$ are hyperimaginary units (hyperspectral units) with the commutative law of multiplication

$$\varepsilon^r\varepsilon^s = \begin{cases} \varepsilon^{r\oplus s\,(\mathrm{mod}\,K)} & \text{if} \quad \varepsilon^K = +1, \\ \mathrm{Hev}\,(l-m)\,\varepsilon^{r\oplus s\,(\mathrm{mod}\,K)} & \text{if} \quad \varepsilon^K = 0, \\ \mathrm{sgn}\,(l-m)\,\varepsilon^{r\oplus s\,(\mathrm{mod}\,K)} & \text{if} \quad \varepsilon^K = -1; \end{cases}$$

$$\mathrm{sgn}\,(x) = \begin{cases} +1, & x \geq 0, \\ -1, & x < 0; \end{cases} \qquad \mathrm{Hev}\,(x) = \begin{cases} +1, & x \geq 0, \\ \ \ 0, & x < 0, \end{cases}$$

where $l \oplus m$ is addition modulo K and the sign and Heaviside functions are used.

We rely on the following hypotheses:

1. The brain interprets each pixel in an image not as a multidimensional vector but as a multidimensional hypercomplex number. If we admit the vector nature of pixels, then we can go even further and assume the possibility of multiplication of these vectors. Thus, we do not refute the vector nature of the pixel, but we enrich it with additional mathematical capabilities. We are talking about expanding the capabilities of the mathematical language for describing the reality by introducing the pixel multiplication operation.

2. The visual systems of animals with different evolutionary histories use different hypercomplex algebras to process color and multichannel images. Apparently, the visual cortex region has the ability to operate with image pixels as with hypercomplex numbers.

3. The brain uses different algebras on the two levels: commutative algebras at the retinal level for image processing and noncommutative algebras at the VC level for image recognition.

We know that animals are able to recognize their surroundings almost instantly and effectively. It is important for an engineer to describe this phenomenon in mathematical terms in order to construct a technical system capable of working as efficiently as the original biological system; the question of whether the mathematical model used is adequate recedes into the background for an engineer (in contrast to a biologist and a physicist).

We disagree with L. Kronecker, who said that "God made the integers, and all else is the work of man." We assume that God was the first engineer, who knew abstract algebra and used the theory of hypercomplex numbers to design visual systems of living organisms.

1. ALGEBRAIC MODELS OF GRAY AND BICHROMATIC IMAGES

Bichromatic 2D images $\mathbf{f}(x_1, x_2) = (f_0(x_1, x_2), f_1(x_1, x_2))\colon \mathbf{R}^2 \to \mathbf{V}^2$ have two attributes: $\mathbf{R}^2$ and $\mathbf{V}^2$, i.e., the physical and visual spaces. We equip these spaces with the structures of 2D algebras of generalized complex numbers $\mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R} \mid 1, I)$ and $\mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R} \mid 1, J)$, respectively; i.e.,

$$\mathbf{R}^2 \to \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}|1,\, I) := \mathbf{R} + \mathbf{R}\,I = \{\mathbf{z} = x_1 + I\,x_2 \,|\, x_1, x_2 \in \mathbf{R}\},$$

$$\mathbf{V}^2 \to \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}|1,\ J) := \mathbf{R} + \mathbf{R}\,J = \{\mathrm{Z} = r + J\,g\,|\,r, g \in \mathbf{R}\},$$

where I and J are the spatial and bichromatic imaginary units, respectively. These algebras are called the *spatial* and *bichromatic algebras* (see [7–9]) of the physical $\mathbf{R}^2$ and visual (perceptual) $\mathbf{V}^2$ spaces, respectively. There exist three spatial algebras.

1. If $I^2 \equiv I_-^2 = -1$, then the algebra $\mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1,\ I_-) = \{\mathbf{z} = x + I_- y\,|\,x, y \in \mathbf{R};\ I_-^2 = -1\}$ is the field of spatial complex numbers, where $I_- = i$ is the usual classical (elliptic) imaginary unit.

2. If $I^2 \equiv I_+^2 = +1$, then the algebra $\mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1,\ I_+) = \{\mathbf{z} = x + I_+ y\,|\,x, y \in \mathbf{R};\ I_+^2 = +1\}$ is the ring of binary spatial complex numbers, where $I_+ = e$ is the classical double (hyperbolic) imaginary unit.

3. If $I^2 \equiv I_0^2 = 0$, then the algebra $\mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1,\ I_0) = \{\mathbf{z} = x + I_0 y\,|\,x, y \in \mathbf{R};\ I_0^2 = -1\}$ is the ring of dual spatial complex numbers, where $I_0 \equiv \varepsilon$ is the classical dual (parabolic) imaginary unit.

It is known (see [10]) that these algebras generate various metrics (Euclidean, Minkowskian, and Galilean) in the 2D physical space. This makes it possible to simply describe geometric transformations of images of various geometric nature in the algebraic language.

Using the algebras $\mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I)$, we can introduce algebraic models of gray images in the form of a function of one generalized complex variable $\mathbf{f}(x + Iy)\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I) \to \mathbf{R}$. There exist three types of such images:

$$(1)\ {}^{-}\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I_-) \to \mathbf{R},\ \ (2)\ {}^{0}\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I_0) \to \mathbf{R},\ \ (3)\ {}^{+}\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I_+) \to \mathbf{R}.$$

These types are defined in the Euclidean (elliptic), Minkowskian (hyperbolic), and Galilean (parabolic) planes, respectively.

Similarly, there exist three perceptual algebras with different geometric (metric) properties.

1. If $J^2 \equiv J_-^2 = -1$, then the perceptual algebra $\mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J_-) = \{\mathbf{z} = x + J_- y\,|\,x, y \in \mathbf{R};$ $J_-^2 = -1\}$ is the field of complex bichromatic numbers, where J_- is the bichromatic imaginary unit similar to the usual classical imaginary unit $J_- \approx i$.

2. If $J^2 \equiv J_+^2 = +1$, then the perceptual algebra $\mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J_+) = \{\mathbf{z} = x + J_+ y\,|\,x, y \in \mathbf{R};$ $J_+^2 = +1\}$ is the ring of double bichromatic numbers, where J_+ is the bichromatic imaginary init similar to the usual double unit $J_+ \approx e$.

3. If $J^2 \equiv J_0^2 = 0$, then the perceptual algebra $\mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J_0) = \{\mathbf{z} = x + J_0 y\,|\,x, y \in \mathbf{R};$ $J_0^2 = 0\}$ is the ring of dual bichromatic numbers, where J_0 is the bichromatic imaginary unit similar to the usual dual unit $J_0 \approx \varepsilon$.

Thus, there exist nine models of bichromatic images $\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I) \to \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J)$ presented below:

$$\begin{aligned}
&{}^{-,-}\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I_-) \to \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J_-), \qquad &&{}^{-,0}\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I_-) \to \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J_0),\\
&{}^{0,-}\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I_0) \to \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J_-), \qquad &&{}^{0,0}\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I_0) \to \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J_0),\\
&{}^{+,-}\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I_+) \to \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J_-), \qquad &&{}^{+,0}\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I_+) \to \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J_0),
\end{aligned}$$

$$\begin{aligned}
&{}^{-,+}\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I_-) \to \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J_+),\\
&{}^{0,+}\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I_0) \to \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J_+),\\
&{}^{+,+}\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1, I_+) \to \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1, J_+).
\end{aligned}$$

To denote six (three spatial and three perceptual) algebras, we use the symbol $\mathrm{Alg}_2^{\mathrm{Ret}}(\mathbf{R}\,|\,1,B)$ or, briefly, $\mathrm{Alg}_2^{\mathrm{Ret}}$, where $\mathrm{Alg}_2^{\mathrm{Ret}}(\mathbf{R}\,|\,1,B) = \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1,I)$ if $B = I$ and $\mathrm{Alg}_2^{\mathrm{Ret}}(\mathbf{R}\,|\,1,B) = \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1,J)$ if $B = J$.

The algebras $\mathrm{Alg}_2^{\mathrm{Ret}}$ have the conjugation operation, which maps each element $Z = a + Bb$ to the element $\bar{Z} = \overline{a + Bb} = a - Bb$. Let $Z = a + Bb$; then the quadratic form $N(Z) := \|Z\| = Z\,\bar{Z} = a^2 - B^2b^2$ is called the *pseudonorm* of the number $Z = a + Bb$, whereas $|Z| = \sqrt{N(Z)} = \sqrt{Z\,\bar{Z}}$ is called its *modulus*. Obviously, $N(Z_1\,Z_2) = N(Z_1)\,N(Z_2)$. Therefore, the 2D algebras $\mathrm{Alg}_2^{\mathrm{Sp}} \equiv \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1,I)$ and $\mathrm{Alg}_2^{\mathrm{Vis}} \equiv \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1,J)$ transform into pseudometric spaces:

$$\mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1,J) \to \mathbf{Geo}_2^{\mathrm{Sp}(s_1,s_2)} = \left\langle \mathbf{R}^2, \rho(\mathbf{z}_1,\mathbf{z}_2)\right\rangle,$$

$$\mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1,J) \to \mathbf{Geo}_2^{\mathrm{Vis}(s_1,s_2)} = \left\langle \mathbf{V}^2, \rho(Z_1,Z_2)\right\rangle,$$

if we define the following pseudometrics in them:

$$\rho(Z_1,Z_2) := \sqrt{(Z_2 - Z_1)\,\overline{(Z_2 - Z_1)}} = \begin{cases} \sqrt{(a_2-a_1)^2 + (b_2-b_1)^2}, & Z \in \mathrm{Alg}_2^{\mathrm{Ret}}(\mathbf{R}\,|\,B_-),\\ \sqrt{(a_2-a_1)^2 - (b_2-b_1)^2}, & Z \in \mathrm{Alg}_2^{\mathrm{Ret}}(\mathbf{R}\,|\,B_+),\\ |a_2 - a_1|, & Z \in \mathrm{Alg}_2^{\mathrm{Ret}}(\mathbf{R}\,|\,B_0), \end{cases}$$

where $Z_1 = a_1 + Bb_1$, $Z_2 = a_2 + Bb_2$, and two symbols ($s_1 = +1$, $s_2 = -1,0,+1$) in the expressions $\mathbf{Geo}_2^{\mathrm{Sp}(s_1,s_2)}$ and $\mathbf{Geo}_2^{\mathrm{Vis}(s_1,s_2)}$ denote the signature of spaces. Depending on the signature (s_1,s_2), the algebras transform into the following pseudometric spaces $\mathbf{Geo}_2^{\mathrm{Ret}(s_1,s_2)}$:

- the two-dimensional Euclidean geometry $\mathbf{Geo}_2^{\mathrm{Ret}(+,+)} = \left\langle \mathrm{Alg}_2^{\mathrm{Ret}}(\mathbf{R}\,|\,B_+);\ \rho\right\rangle$ (the spatial $\mathbf{Geo}_2^{\mathrm{Sp}(+,+)}$ and perceptual $\mathbf{Geo}_2^{\mathrm{Vis}(+,+)}$ geometries);
- the two-dimensional Minkowskian geometry $\mathbf{Geo}_2^{\mathrm{Ret}(+,-)} = \left\langle \mathrm{Alg}_2^{\mathrm{Ret}}(\mathbf{R}\,|\,B_-);\ \rho\right\rangle$ (the spatial $\mathbf{Geo}_2^{\mathrm{Sp}(+,-)}$ and perceptual $\mathbf{Geo}_2^{\mathrm{Vis}(+,-)}$ geometries);
- the two-dimensional Galilean geometry $\mathbf{Geo}_2^{\mathrm{Ret}(+,0)} = \left\langle \mathrm{Alg}_2^{\mathrm{Ret}}(\mathbf{R}\,|\,B_0);\ \rho\right\rangle$ (the spatial $\mathbf{Geo}_2^{\mathrm{Sp}(+,0)}$ and perceptual $\mathbf{Geo}_2^{\mathrm{Vis}(+,0)}$ geometries).

The set of all points of the generalized complex plane $\mathbf{Geo}_2^{\mathrm{Ret}(s_1,s_2)}$ satisfying the equation $|Z|^2 = a^2 - B^2b^2 = R^2$ is called the $\mathbf{Geo}_2^{\mathrm{Ret}(s_1,s_2)}$-*circle of radius R centered at the origin.* Let $\mathrm{Alg}_2^{\mathrm{Ret}}(\mathbf{R}\,|\,1,B) \equiv \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1,I)$; then there are three types of circles: the classical (Euclidean) $\mathbf{Geo}_2^{\mathrm{Sp}(+,+)}$-circle, the Minkowskian (hyperbolic) $\mathbf{Geo}_2^{\mathrm{Sp}(+,-)}$-circle, and the (Galilean) $\mathbf{Geo}_2^{Sp(+,0)}$-circle (i.e., two parallel lines).

Let $Z = a + Bb$ be an arbitrary generalized complex number (spatial or bichromatic). Then the number $Z_0 = Z/|Z|$ has the unit modulus if $|Z| = R \neq 0$. Clearly, $Z = |Z|\,(a/|Z| + Bb/|Z|) = R\,(\mathbf{cos}\,\alpha + B\,\mathbf{sin}\,\alpha) = R\,e^{B\theta}$, where $\mathbf{cos}\,\alpha$ and $\mathbf{sin}\,\alpha$ are trigonometric Euclidean (the classical functions $\mathbf{cos}\,\alpha = \cos\alpha$ and $\mathbf{sin}\,\alpha = \sin\alpha$), Minkowskian (the hyperbolic functions $\mathbf{cos}\,\alpha = \cosh\alpha$ and $\mathbf{sin}\,\alpha = \sinh\alpha$), and Galilean functions ($\mathbf{cos}\,\alpha = \mathrm{cg}\,\alpha$ and $\mathbf{sin}\,\alpha = \mathrm{sg}\,\alpha$).

Definition 1. A *bichromatic image*

$$\mathbf{f}(\mathbf{z})\colon \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1,I) \to \mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}\,|\,1,J)$$

is an $\mathrm{Alg}_2^{\mathrm{Vis}}(\mathbf{R}|J)$-valued function depending on the complex variable $\mathbf{z} \in \mathrm{Alg}_2^{\mathrm{Sp}}(\mathbf{R}\,|\,1,I)$:

$$\mathbf{f}(\mathbf{z}) = f_0(x_1 + I\,x_2) + J f_1(x_1 + I\,x_2). \tag{1}$$

Definition 2. Transformations

$$\mathbf{z}' = \mathbf{z} + \mathbf{w}, \quad \mathbf{z}' = \lambda \mathbf{z}, \quad \mathbf{z}' = e^{I\varphi_{\rm sp}}\mathbf{z}; \quad Z' = Z + W, \quad Z' = \mu Z, \quad Z' = e^{J\theta_{\rm ch}} Z,$$

where $\mathbf{z}, \mathbf{z}', \mathbf{w} \in \mathrm{Alg}_2^{\rm Sp}$ and $Z, Z, W \in \mathrm{Alg}_2^{\rm Vis}$, are called *a translation, a scaling*, and *a rotation* of the physical $\mathbf{Geo}_2^{\mathrm{Sp}(s_1,s_2)}$ and bichromatic $\mathbf{Geo}_2^{\mathrm{Vis}(s_1,s_2)}$ spaces, respectively.

These transformations form:

(i) two groups of spatial $\mathbf{Tr}(\mathbf{Geo}_2^{\mathrm{Sp}(s_1,s_2)})$ and bichromatic $\mathbf{Tr}(\mathbf{Geo}_2^{\mathrm{Vis}(s_1,s_2)})$ translations,

(ii) two groups of spatial $\mathbf{Sc}(\mathbf{Geo}_2^{\mathrm{Sp}(s_1,s_2)})$ and bichromatic $\mathbf{Sc}(\mathbf{Geo}_2^{\mathrm{Vis}(s_1,s_2)})$ scaling transformations,

(iii) two groups of physical $\mathbf{Rot}(\mathbf{Geo}_2^{\mathrm{Sp}(s_1,s_2)})$ and bichromatic $\mathbf{Rot}(\mathbf{Geo}_2^{\mathrm{Vis}(s_1,s_2)})$ rotations.

Image transformations (geometric and color distortions) in the physical and perceptual spaces can be described in terms of spatial and perceptual algebras. These distortions can be caused by spatial transformations (translations $\mathbf{z}' = \mathbf{z} + \mathbf{w}$, rotations $\mathbf{z}' = e^{I\varphi_{\rm sp}}\mathbf{z}$, and scaling transformations $\mathbf{z}' = \lambda\mathbf{z}$) and bichromatic transformations (bichromatic translations $\mathbf{f} + W$, color conversions $e^{J\theta_{\rm ch}}\mathbf{f}$, and saturation transformations $\mu\mathbf{f}$).

If $\mathbf{f}(\mathbf{z})$ is some original bichromatic image, then the image

$$^{\mu,\theta_{\rm ch},W}\mathbf{f}_{\lambda,\varphi_{\rm sp},\mathbf{w}}(\mathbf{z}) = \mu\, e^{J\theta_{\rm ch}}\, \mathbf{f}(\lambda e^{J\varphi_{\rm sp}}\mathbf{z} + \mathbf{w}) + W \tag{2}$$

is its distorted version. The spatial distortions here are caused by transformations of the physical space: $\mathbf{z} \to \lambda e^{J\varphi_{\rm sp}}\mathbf{z} + \mathbf{w}$, while the color distortions are due to transformations of the perceptual space $\mathbf{f} \to \mu e^{J\theta_{\rm ch}}\mathbf{f} + W$.

It turns out that the algebraic models of the bichromatic image (1) and its distorted version (2) are universal in the sense that many types of multichannel images defined in Euclidean and non-Euclidean spaces can be written in a similar form using hypercomplex algebras. This circumstance makes it possible to develop new algorithms for image processing and invariant image recognition.

2. ALGEBRAIC MODELS OF COLOR IMAGES

A color image is a vector-valued function of the form $\mathbf{f}(\mathbf{x})\colon \mathbf{R}^n \to \mathbf{V}^3_{\rm rgb}$, where $\mathbf{V}^3_{\rm rgb}$ is the trichromatic (color) RGB-space. We will interpret it as a triplet-valued signal $\mathbf{f}(\mathbf{x}) = f_r(\mathbf{x})1 + f_g(\mathbf{x})\varepsilon^1_{\rm col} + f_b(\mathbf{x})\varepsilon^2_{\rm col}$, which takes values in the triplet (color) algebra $\mathrm{Alg}_3^{\rm Vis}(\mathbf{R}\,|\,1,\varepsilon,\varepsilon^2) := \mathbf{R}1_{\rm col} + \mathbf{R}\varepsilon^1_{\rm col} + \mathbf{R}\varepsilon^2_{\rm col}$, where $1_{\rm col}$, $\varepsilon^1_{\rm col}$, and $\varepsilon^2_{\rm col}$ are three hyperimaginary (color) units with one of the three properties: $\varepsilon^3_{\rm col} = +1$, $\varepsilon^3_{\rm col} = 0$, or $\varepsilon^3_{\rm col} = -1$ (see [11–16]). For brevity, we will denote them as 1, ε^1, and ε^2. Obviously, there exist three perceptual color algebras.

1. If $\varepsilon^3 = \varepsilon^3_- = -1$, then $\mathrm{Alg}_3^{\rm Vis}(\mathbf{R}\,|\,1,\varepsilon_-,\varepsilon^2_-) := \mathbf{R}1 + \mathbf{R}\varepsilon^1_- + \mathbf{R}\varepsilon^2_- = \{\mathcal{C} = r\,1 + g\,\varepsilon^1_- + b\,\varepsilon^2_- \,|\, r,g,b \in \mathbf{R}\}$ is the color algebra of color acyclic numbers.

2. If $\varepsilon^3 = \varepsilon^3_+ = +1$, then $\mathrm{Alg}_3^{\rm Vis}(\mathbf{R}\,|\,1,\varepsilon_+,\varepsilon^2_+) := \mathbf{R}1 + \mathbf{R}\varepsilon^1_+ + \mathbf{R}\varepsilon^2_+ = \{\mathcal{C} = r\,1 + g\,\varepsilon^1_+ + b\,\varepsilon^2_+ \,|\, r,g,b \in \mathbf{R}\}$ is the color algebra of color cyclic numbers.

3. If $\varepsilon^3 = \varepsilon^3_0 = 0$, then $\mathrm{Alg}_3^{\rm Vis}(\mathbf{R}\,|\,1,\varepsilon_0,\varepsilon^2_0) := \mathbf{R}1 + \mathbf{R}\varepsilon^1_0 + \mathbf{R}\varepsilon^2_0 = \{\mathcal{C} = r1 + g\,\varepsilon^1_0 + b\,\varepsilon^2_0 \,|\, r,g,b \in \mathbf{R}\}$ is the color algebra of color nilpotent numbers.

Color cyclic numbers of the form $\mathcal{C} = x\,1 + y\,\varepsilon + z\,\varepsilon^2$ ($\varepsilon^3 = 1$) were first discovered by Greaves in [11]. He called these numbers *triplets*. Taking into account the context of this paper, we call them *color numbers*. The addition and multiplication of two color numbers $\mathcal{C}_1 = (r_1 + g_1\varepsilon + b_1\varepsilon^2)$ and $\mathcal{C}_2 = (r_2 + g_2\varepsilon + b_2\varepsilon^2)$ are defined as:

$$\mathcal{C}_1 + \mathcal{C}_2 = (r_1 + g_1\varepsilon + b_1\varepsilon^2) + (r_2 + g_2\varepsilon + b_2\varepsilon^2) = (r_1 + r_2) + (g_1 + g_2)\,\varepsilon + (b_1 + b_2)\,\varepsilon^2,$$

$$\mathcal{C}_1 \cdot \mathcal{C}_2 = (r_1 + g_1\varepsilon + b_1\varepsilon^2)(r_2 + g_2\varepsilon + b_2\varepsilon^2)$$
$$= (r_1r_2 + g_1b_2 + b_1g_2) + (r_1g_2 + r_2g_1 + b_1b_2)\,\varepsilon + (r_1b_2 + g_1g_2 + r_2b_1)\,\varepsilon^2.$$

It is easy to verify that the triplet product is isomorphic to the cyclic convolution

$$\mathcal{C}_1\mathcal{C}_2 = (r_1 + g_1\varepsilon + b_1\varepsilon^2)(r_2 + g_2\varepsilon + b_2\varepsilon^2) \equiv (r_1, g_1, b_1) * (r_2, g_2, b_2)$$
$$= (r_1r_2 + g_1b_2 + b_1g_2,\ r_1g_2 + r_2g_1 + b_1b_2,\ r_1b_2 + g_1g_2 + r_2b_1).$$

The triplet conjugation of a number $\mathcal{C} = (r + g\varepsilon + b\varepsilon^2)$ is described by the equality

$$\bar{\mathcal{C}} = \overline{r + g\varepsilon + b\varepsilon^2} = r + g\varepsilon^2 + b\varepsilon^1.$$

The norm $\|\mathcal{C}\|_2$ and the modulus $|\mathcal{C}|_2$ are defined by the expressions

$$\|\mathcal{C}\|_2 = \mathcal{C}\bar{\mathcal{C}} = (r^2 + g^2 + b^2) - (rg + rb + gb), \quad |\mathcal{C}|_2 = \sqrt{\|\mathcal{C}\|_2} \equiv \sqrt{\mathcal{C}\bar{\mathcal{C}}}.$$

Greaves (see [11]) showed that each triplet number has three norms:

$$\begin{gathered} ||\mathcal{C}||_1 = |r + g + b|, \quad ||\mathcal{C}||_2 = (r^2 + g^2 + b^2) - (rg + rb + gb), \\ ||\mathcal{C}||_3 = ||\mathcal{C}||_1\,||\mathcal{C}||_2 \equiv r^3 + g^3 + b^3 - 3rgb. \end{gathered} \tag{3}$$

If the distance $\rho(\mathcal{C}, \mathcal{D})$ between two triplet numbers $\mathcal{C}$ and $\mathcal{D}$ is defined as the modulus of their difference $\mathcal{C} - \mathcal{D} = \mathcal{U} = r + g\varepsilon + b\varepsilon^2$, then we can introduce three metrics in the color perceptual space:

$$\rho_1(\mathcal{C}, \mathcal{D}) = |\mathcal{C} - \mathcal{D}|_1 = |\mathcal{U}|_1 = |r + g + b|,$$
$$\rho_2(\mathcal{C}, \mathcal{D}) = |\mathcal{C} - \mathcal{D}|_2 = |\mathcal{U}|_2 = \sqrt[2]{(r^2 + g^2 + b^2) - (rg + rb + gb)},$$
$$\rho_3(\mathcal{C}, \mathcal{D}) = |\mathcal{C} - \mathcal{D}|_3 = |\mathcal{U}|_3 = \sqrt[3]{r^3 + g^3 + b^3 - 3rgb}.$$

Consequently, the algebra of color numbers $\mathrm{Alg}_3^{\mathrm{Vis}}(\mathbf{R}\,|\,1, \varepsilon, \varepsilon^2)$ turns into three 3D metric spaces:

$$\mathbf{Geo}_3^{\mathrm{Vis1}} = \left\langle\left\langle \mathrm{A}_3(\mathbf{R}\,|\,1, \varepsilon, \varepsilon^2)\,\middle|\,|r + g + b| \right\rangle\right\rangle,$$
$$\mathbf{Geo}_3^{\mathrm{Vis2}} = \left\langle\left\langle \mathrm{A}_3(\mathbf{R}\,|\,1, \varepsilon, \varepsilon^2)\,\middle|\,\sqrt[2]{(r^2 + g^2 + b^2) - (rg + rb + gb)} \right\rangle\right\rangle,$$
$$\mathbf{Geo}_3^{\mathrm{Vis3}} = \left\langle\left\langle \mathrm{A}_3(\mathbf{R}\,|\,1, \varepsilon, \varepsilon^2)\,\middle|\,\sqrt[3]{r^3 + g^3 + b^3 - 3rgb} \right\rangle\right\rangle.$$

Greaves gave algebraic and geometric interpretations of triplet numbers. Algebraically, a color number $\mathcal{C} = x + y\varepsilon + z\varepsilon^2$ is the point $\mathcal{C} = \mathcal{C}(x, y, z) \in \mathbf{V}^3_{\mathrm{RGB}}$ in the 3D color space $\mathbf{V}^3_{\mathrm{RGB}}$ with coordinates (x, y, z). The color algebra is algebraically the direct sum of the field of real numbers $\mathbf{R}$ and the field of complex numbers $\mathrm{Alg}_3^{\mathrm{Vis}} = \mathbf{R}e_{\mathrm{lu}} + \mathbf{C}\,\mathbf{E}_{\mathrm{ch}} = \mathbf{R} \oplus \mathbf{C}$, where $\mathbf{e}_{\mathrm{lu}} = (1 + \varepsilon + \varepsilon^2)/3$ and $\mathbf{E}_{\mathrm{ch}} = (1 + \omega_3\varepsilon^2 + \omega_3^2\varepsilon)/3$ are the so-called orthogonal idempotents (projectors) $\mathbf{e}_{\mathrm{lu}}^2 = \mathbf{e}_{\mathrm{lu}}$, $\mathbf{E}_{\mathrm{ch}}^2 = \mathbf{E}_{\mathrm{ch}}$, $\mathbf{e}_{\mathrm{lu}}\mathbf{E}_{\mathrm{ch}} = \mathbf{E}_{\mathrm{ch}}\mathbf{e}_{\mathrm{lu}} = 0$, and $\omega_3 := \exp(2\pi/3)$. Indeed, in accordance with the polynomial Chinese remainder theorem, we have

$$\mathrm{Alg}_3^{\mathrm{Vis}} \approx \mathbf{R}[x]/(x^3 - 1) = \mathbf{R}[x]/(x - 1)(x^2 + x + 1) \approx \mathbf{R}[x]/(x - 1) \oplus \mathbf{R}[x]/(x^2 + x + 1) \approx \mathbf{R} \oplus \mathbf{C}.$$

Consequently, every color number $\mathcal{C} = x + y\varepsilon + z\varepsilon^2$ is a linear combination $\mathcal{C} = a_{\text{lu}}\mathbf{e}_{\text{lu}} + Z_{\text{ch}}\mathbf{E}_{\text{ch}} = (a_{\text{lu}}, Z_{\text{ch}})$ of the real $a_{\text{lu}}\mathbf{e}_{\text{lu}}$ and complex $Z_{\text{ch}}\mathbf{E}_{\text{ch}}$ components in the idempotent basis $\{\mathbf{e}_{\text{lu}}, \mathbf{E}_{\text{ch}}\}$, where

$$a_{\text{lu}}\mathbf{e}_{\text{lu}} = \mathcal{C}\mathbf{e}_{\text{lu}} = \mathbf{e}_{\text{lu}} = (r + g\varepsilon + b\varepsilon^2)[(1+\varepsilon+\varepsilon^2)/3] = (r+g+b)[(1+\varepsilon+\varepsilon^2)/3],$$

$$Z_{\text{ch}}\mathbf{E}_{\text{ch}} = \mathcal{C}\mathbf{E}_{\text{ch}} = \mathbf{e}_{\text{lu}} = (r + g\varepsilon + b\varepsilon^2)[(1+\omega_3\varepsilon^2+\omega_3^2\varepsilon)/3] = (r+g\omega_3^1+b\omega_3^2)[(1+\omega_3\varepsilon^2+\omega_3^2\varepsilon)/3]$$

and, hence,

$$a_{\text{lu}} = (r+g+b), \quad Z_{\text{ch}} = (r+g\omega_3^1+b\omega_3^2) = (r-(g+b)/2) + i\sqrt{3}\,(g-b)/2.$$

We call $a_{\text{lu}} \in \mathbf{R}$ and $Z_{\text{ch}} \in \mathbf{C}$ the *luminance* and *chromatic numbers*, respectively. For this reason, the color images can be considered in two formats:

$$\mathbf{f}(\mathbf{z}) = f_R(\mathbf{z})1 + f_G(\mathbf{z})\varepsilon + f_B(\mathbf{z})\varepsilon^2, \quad \mathbf{f}(\mathbf{z}) = f_{\text{lu}}(\mathbf{z})\mathbf{e}_{\text{lu}} + \mathbf{f}_{\text{ch}}(\mathbf{z})\mathbf{E}_{\text{ch}} = (f_{\text{lu}}(\mathbf{z}), \mathbf{f}_{\text{ch}}(\mathbf{z})).$$

The first representation is called the (R,G,B) format, and the second is the luminance–chrominance (LC) format. The latter format $(f_{\text{lu}}(\mathbf{z}), \mathbf{f}_{\text{ch}}(\mathbf{z}))$ defines an image in terms of the luminance $f_{\text{lu}}(\mathbf{z})$ and chromatic $\mathbf{f}_{\text{ch}}(\mathbf{z})$ components, where $|\mathbf{f}_{\text{ch}}(\mathbf{z})|$ is the saturation and $\mathbf{arg}\{\mathbf{f}_{\text{ch}}(\mathbf{z})\}$ is the hue of $\mathbf{f}(\mathbf{z})$.

Changes in the hue, saturation, and luminance are fairly easy to describe in terms of the color algebra. Let, for example, $\mathrm{A} = (a_{\text{lu}}, Z_{\text{ch}}) = (a_{\text{lu}}, |Z_{\text{ch}}|e^{i\varphi})$, where $a_{\text{lu}} > 0$. Then the transformations

$$\mathbf{f}(\mathbf{z}) \to \mathrm{A}\cdot\mathbf{f}(\mathbf{z})$$
$$= (a_{\text{lu}}, Z_{\text{ch}})\,(f_{\text{lu}}(\mathbf{z}), \mathbf{f}_{\text{ch}}(\mathbf{z})) = (a_{\text{lu}}, |Z_{\text{ch}}|e^{i\varphi})\,(f_{\text{lu}}(\mathbf{z}), \mathbf{f}_{\text{ch}}(\mathbf{z})) = (a_{\text{lu}}f_{\text{lu}}(\mathbf{z}), |Z_{\text{ch}}|e^{i\varphi}\mathbf{f}_{\text{ch}}(\mathbf{z}))$$

change the luminance, hue, and saturation of a color image. The set of such transformations forms the luminance–chromatic group

$$\mathbf{LCG}(\mathrm{Alg}_3^{\mathrm{Vis}}(\mathbf{R}\,|\,\varepsilon)) = \{(a_{\text{lu}}, Z_{\text{ch}})\,|\,(a_{\text{lu}} \in \mathbf{R}^+)\,\&\,(Z_{\text{ch}} \in \mathbf{C})\}.$$

In particular,

- if $\mathrm{A} = (a_{\text{lu}}, Z_{\text{ch}}) = (1, e^{i\varphi})$, then the transformations

$$\mathbf{f}(\mathbf{z}) \to \mathrm{A}\cdot\mathbf{f}(\mathbf{z}) = (1, e^{i\varphi})\,(f_{\text{lu}}(\mathbf{z}), \mathbf{f}_{\text{ch}}(\mathbf{z})) = (f_{\text{lu}}(\mathbf{z}), e^{i\varphi}\mathbf{f}_{\text{ch}}(\mathbf{z}))$$

change the hue of the image. The set of such transformations forms the orthogonal group of transformations of the hue $\mathbf{HOG}(\mathrm{Alg}_3^{\mathrm{Vis}}(\mathbf{R}\,|\,\varepsilon)) = \{(1, e^{i\varphi})\,|\,e^{i\varphi} \in \mathbf{C}\}$;

- let $\mathrm{A} = (1, s)$, $s > 0$; then the transformations

$$\mathbf{f}(\mathbf{z}) \to \mathrm{A}\cdot\mathbf{f}(\mathbf{z}) = (1, s)\,(f_{\text{lu}}(\mathbf{z}), \mathbf{f}_{\text{ch}}(\mathbf{z})) = (f_{\text{lu}}(\mathbf{z}), s\mathbf{f}_{\text{ch}}(\mathbf{z}))$$

change the saturation of the original image. The set of such transformations forms the group of transformations of the saturation $\mathbf{SaG}(\mathrm{Alg}_3^{\mathrm{Vis}}(\mathbf{R}\,|\,\varepsilon_{\text{col}})) = \{(1, s)\,|\,s \in \mathbf{R}^+\}$;

- if $\mathrm{A} = (1, Z_{\text{ch}}) = (1, se^{i\varphi})$, then the transformations

$$\mathbf{f}(\mathbf{z}) \to \mathrm{A}\cdot\mathbf{f}(\mathbf{z}) = (1, se^{i\varphi})\,(f_{\text{lu}}(\mathbf{z}), \mathbf{f}_{\text{ch}}(\mathbf{z})) = (f_{\text{lu}}(\mathbf{z}), se^{i\varphi}\mathbf{f}_{\text{ch}}(\mathbf{z}))$$

change both the hue and the saturation of the original image. The set of such transformations forms the chromatic group

$$\mathbf{ChG}(\mathrm{Alg}_3^{\mathrm{Vis}}(\mathbf{R}\,|\,\varepsilon_{\text{col}})) = \{(1, se^{i\varphi})\,|\,(e^{i\varphi} \in \mathbf{C})\,\&\,(s \in \mathbf{R}^+)\}.$$

3. MULTIPLET MODELS OF MULTICHANNEL IMAGES

Traditionally, multichannel images are interpreted as KD vector-valued signals $\mathbf{f}(\mathbf{x}) = (f_0(\mathbf{x}), f_1(\mathbf{x}), \ldots, f_{K-1}(\mathbf{x}))\colon \mathbf{R}^n \to \mathbf{V}^K$. We will interpret them as multiplet-valued signals $\mathbf{f}(\mathbf{x}) = f_0(\mathbf{x}) + f_1(\mathbf{x})\varepsilon^1 + f_2(\mathbf{x})\varepsilon^2 + \ldots + f_{K-1}(\mathbf{x})\varepsilon^{K-1}$, which take value in the multiplet algebra $\mathrm{Alg}_K^{\mathrm{Vis}}(\mathbf{R}\,|\,1, \varepsilon, \ldots, \varepsilon^{K-1}) = \mathbf{R}1 + \mathbf{R}\varepsilon^1 + \ldots + \mathbf{R}\varepsilon^{K-1}$, where $\mathbf{x} \in \mathbf{R}^n$ and $1, \varepsilon^1, \ldots, \varepsilon^{K-1}$ ($\varepsilon^K = +1,\ 0,\ -1$) are multicolor hyperimaginary units (see [12–16]).

Any multiplet number can be represented by a linear combination of hyperimaginary units $\mathrm{M} = \sum_{i=0}^{K-1} a_i\varepsilon^i$ with $a_i \in \mathbf{R}$. These numbers (depending on $\varepsilon^K = +1, 0, -1$) form three multiplet algebras:

$$\mathrm{Alg}_K^{+,\mathrm{Vis}}(\mathbf{R}) \equiv \sum_{i=0}^{K-1} \mathbf{R}\,\varepsilon_+^i, \quad \mathrm{Alg}_K^{-,\mathrm{Vis}}(\mathbf{R}) \equiv \sum_{i=0}^{K-1} \mathbf{R}\,\varepsilon_-^i, \quad \mathrm{Alg}_K^{0,\mathrm{Vis}}(\mathbf{R}) \equiv \sum_{i=0}^{K-1} \mathbf{R}\,\varepsilon_0^i.$$

The addition of multiplet numbers M_1 and M_2 is implemented in the three algebras componentwise:

$$\mathrm{M} = \mathrm{M}_1 + \mathrm{M}_2 = \sum_{i=0}^{K-1} a_i\varepsilon^i + \sum_{i=0}^{K-1} b_i\varepsilon^i = \sum_{i=0}^{K-1} (a_i + b_i)\varepsilon^i.$$

Consequently, with respect to addition, all three algebras form the same vector space. The multiplication rules for any pair of multiplet numbers M_1 and M_2 are different in the three multiplet algebras:

$$\mathrm{M} = \mathrm{M}_1 \cdot \mathrm{M}_2 = \Bigl(\sum_{n=0}^{K-1} a_n\varepsilon_+^n\Bigr)\Bigl(\sum_{m=0}^{K-1} b_m\varepsilon_+^m\Bigr) = \sum_{l=0}^{K-1}\Bigl(\sum_{m=0}^{K-1} a_{l\ominus m}b_m\Bigr)\varepsilon_+^l = \sum_{l=0}^{K-1} c_l\varepsilon_+^l \quad \text{for} \quad \mathrm{Alg}_K^{+,\mathrm{Vis}}(\mathbf{R}),$$

$$\mathrm{M} = \mathrm{M}_1 \cdot \mathrm{M}_2 = \Bigl(\sum_{n=0}^{K-1} a_n\varepsilon_-^n\Bigr)\Bigl(\sum_{m=0}^{K-1} b_m\varepsilon_-^m\Bigr) = \sum_{l=0}^{K-1}\Bigl(\sum_{m=0}^{K-1} \mathrm{sgn}\,(l-m) a_{l\ominus m}b_m\Bigr)\varepsilon_-^l$$

$$= \sum_{l=0}^{K-1} c_l\varepsilon_-^l \quad \text{for} \quad \mathrm{Alg}_K^{-,\mathrm{Vis}}(\mathbf{R}),$$

$$\mathrm{M} = \mathrm{M}_1 \cdot \mathrm{M}_2 = \Bigl(\sum_{n=0}^{K-1} a_n\varepsilon_0^n\Bigr)\Bigl(\sum_{m=0}^{K-1} b_m\varepsilon_0^m\Bigr) = \sum_{l=0}^{K-1}\Bigl(\sum_{m=0}^{K-1} \mathrm{Hev}\,(l-m) a_{l\ominus m}b_m\Bigr)\varepsilon_0^l$$

$$= \sum_{l=0}^{K-1} c_l\varepsilon_0^l \quad \text{for} \quad \mathrm{Alg}_K^{0,\mathrm{Vis}}(\mathbf{R}).$$

It is easy to see that the multiplet products are isomorphic to discrete K-point convolutions that are the cyclic, acyclic, and nilpotent convolutions, respectively:

$$c_l = \sum_{m=0}^{K-1} a_{l\ominus m}b_m, \quad c_l = \sum_{m=0}^{K-1} \mathrm{sgn}\,(l-m) a_{l\ominus m}b_m, \quad c_l = \sum_{m=0}^{K-1} \mathrm{Hev}\,(l-m) a_{l\ominus m}b_m,$$

where $l \ominus m$ is subtraction modulo K. Using the polynomial Chinese remainder theorem, we can easily prove that the two algebra $\mathrm{Alg}_K^{+,\mathrm{Vis}}(\mathbf{R})$ and $\mathrm{Alg}_K^{-,\mathrm{Vis}}(\mathbf{R})$ are the direct sums of the real and complex fields:

$$\mathrm{Alg}_K^{+,\mathrm{Vis}}(\mathbf{R}) \equiv (\mathbf{R}\cdot\mathbf{e}_{\mathrm{lu}}^1) \oplus (\mathbf{R}\cdot\mathbf{e}_{\mathrm{lu}}^2) \oplus \sum_{j=1}^{K/2-1} \mathbf{C}\cdot\mathbf{E}_{\mathrm{ch}}^j \quad \text{if } K \text{ is even}$$

and

$$\mathrm{Alg}_K^{+,\mathrm{Vis}}(\mathbf{R}) \equiv \mathbf{R}^{K_{\mathrm{lu}}} \oplus \mathbf{C}^{K_{\mathrm{ch}}} = \mathbf{R}\cdot\mathbf{e}_{\mathrm{lu}}^1 \oplus \sum_{j=1}^{(K-1)/2} \mathbf{C}\cdot\mathbf{E}_{\mathrm{ch}}^j \quad \text{if } K \text{ is odd;}$$

similarly,

$$\mathrm{Alg}_K^{-,\mathrm{Vis}}(\mathbf{R}) = \sum_{j=1}^{K/2} \mathbf{C}\cdot\mathbf{E}_{\mathrm{ch}}^j \quad \text{if } K \text{ is even}$$

and

$$\mathrm{Alg}_K^{-,\mathrm{Vis}}(\mathbf{R}) = \mathbf{R}\cdot\mathbf{e}_{\mathrm{lu}}^1 + \sum_{j=1}^{(K-1)/2} \mathbf{C}\cdot\mathbf{E}_{\mathrm{ch}}^j \quad \text{if } K \text{ is odd,}$$

where $\mathbf{e}_{\mathrm{lu}}^i$ and $\mathbf{E}_{\mathrm{ch}}^j$ are "real" and "complex" orthogonal idempotents such that $(\mathbf{e}_{\mathrm{lu}}^i)^2 = \mathbf{e}_{\mathrm{lu}}^i$, $(\mathbf{E}_{\mathrm{ch}}^j)^2 = \mathbf{E}_{\mathrm{ch}}^j$, and $\mathbf{e}_{\mathrm{lu}}^i\mathbf{E}_{\mathrm{ch}}^j = \mathbf{E}_{\mathrm{ch}}^j\mathbf{e}_{\mathrm{lu}}^i$ for all i and j.

Let $K_{\mathrm{lu}} = 0,1,2$ and $K_{\mathrm{ch}} = K/2,\ K/2-1,\ (K-1)/2$. Every multiplet $\mathrm{M} \in \mathrm{Alg}_K^{\pm,\mathrm{Vis}}(\mathbf{R})$ can be presented as a linear combination of K_{lu} "scalar" terms and K_{ch} "complex" terms:

$$\mathrm{M} = \sum_{i=1}^{K_{\mathrm{lu}}} a_i\,\mathbf{e}_{\mathrm{lu}}^i + \sum_{j=1}^{K_{\mathrm{ch}}} \mathbf{z}_j\,\mathbf{E}_{\mathrm{ch}}^j.$$

The real numbers $a_i \in \mathbf{R}$ are called *multiluminances* and the complex numbers $\mathbf{z}_j \in \mathbf{C}$ are called *multichromates*. In this representation, the two basic arithmetic operations have a simple form:

$$\mathrm{M}_1+\mathrm{M}_2=\Big(\sum_{i=1}^{K_{\mathrm{lu}}} a_i\mathbf{e}_{\mathrm{lu}}^i+\sum_{j=1}^{K_{\mathrm{ch}}}\mathbf{z}_j\,\mathbf{E}_{\mathrm{ch}}^j\Big)+\Big(\sum_{i=1}^{K_{\mathrm{lu}}} b_i\mathbf{e}_{\mathrm{lu}}^i+\sum_{j=1}^{K_{\mathrm{ch}}}\mathbf{w}_j\,\mathbf{E}_{\mathrm{ch}}^j\Big)=\Big(\sum_{i=1}^{K_{\mathrm{lu}}}(a_i+b_i)\,\mathbf{e}_{\mathrm{lu}}^i+\sum_{j=1}^{K_{\mathrm{ch}}}(\mathbf{z}_j+\mathbf{w}_j)\mathbf{E}_{\mathrm{ch}}^j\Big),$$

$$\mathrm{M}_1\cdot\mathrm{M}_2=\Big(\sum_{i=1}^{K_{\mathrm{lu}}} a_i\mathbf{e}_{\mathrm{lu}}^i+\sum_{j=1}^{K_{\mathrm{ch}}}\mathbf{z}_j\,\mathbf{E}_{\mathrm{ch}}^j\Big)\cdot\Big(\sum_{i=1}^{K_{\mathrm{lu}}} b_i\mathbf{e}_{\mathrm{lu}}^i+\sum_{j=1}^{K_{\mathrm{ch}}}\mathbf{w}_j\,\mathbf{E}_{\mathrm{ch}}^j\Big)=\Big(\sum_{i=1}^{K_{\mathrm{lu}}} a_ib_i\,\mathbf{e}_{\mathrm{lu}}^i+\sum_{j=1}^{K_{\mathrm{ch}}}\mathbf{z}_j\mathbf{w}_j\,\mathbf{E}_{\mathrm{ch}}^j\Big).$$

Multiplet algebras are not fields. They form numerical rings with zero divisors.

Definition 3. Multichannel 2D signals of the type

$$\mathbf{f}(\mathbf{z}) = \sum_{i=0}^{K-1} f_i(\mathbf{z})\varepsilon^i, \quad \mathbf{f}(\mathbf{z}) = \sum_{i=1}^{K_{\mathrm{lu}}}[f_{\mathrm{lu}}^i(\mathbf{z})\cdot\mathbf{e}_{\mathrm{lu}}^i] + \sum_{j=1}^{K_{\mathrm{ch}}}[\mathbf{f}_{\mathrm{ch}}^j(\mathbf{z})\cdot\mathbf{E}_{\mathrm{ch}}^j]$$

are called *multiplet-valued images* in the multiplet and multiluminance–chrominance formats, respectively.

The first format specifies an image by K luminances of each channel; the second format specifies it by K_{lu} luminance terms $(f_{\mathrm{lu}}^1(\mathbf{z}),\ f_{\mathrm{lu}}^2(\mathbf{z}),\dots,\ f_{\mathrm{lu}}^{K_{\mathrm{lu}}}(\mathbf{z}))$ and K_{ch} chromatic terms $(\mathbf{f}_{\mathrm{ch}}^1(\mathbf{z}),\ \mathbf{f}_{\mathrm{ch}}^2(\mathbf{z}),\dots,$

$\mathbf{f}_{\mathrm{ch}}^{K_{\mathrm{ch}}}(\mathbf{z}))$, where $|\mathbf{f}_{\mathrm{ch}}^1(\mathbf{z})|$, $|\mathbf{f}_{\mathrm{ch}}^2(\mathbf{z})|, \ldots,$ $|\mathbf{f}_{\mathrm{ch}}^{K_{\mathrm{ch}}}(\mathbf{z})|$ are multisaturations and $\arg\{\mathbf{f}_{\mathrm{ch}}^1(\mathbf{z})\}$, $\arg\{\mathbf{f}_{\mathrm{ch}}^2(\mathbf{z})\}$, $\ldots$, $\arg\{\mathbf{f}_{\mathrm{ch}}^{K_{\mathrm{ch}}}(\mathbf{x})\}$ are multihues of the multichannel image $\mathbf{f}(\mathbf{z})$.

It is easy to describe changes in the multiluminance and multichrominance of the multichannel image in terms of the multiplet algebra $\mathrm{Alg}_K^{\mathrm{Vis}}(\mathbf{R})$ as a transformation $\mathbf{f}(\mathbf{z}) \to \mathrm{M}\cdot\mathbf{f}(\mathbf{z})$ for an appropriate multiplet number M. For example, if

$$\mathrm{M} = \big(a_{\mathrm{lu}}^1,\, a_{\mathrm{lu}}^2, \ldots,\, a_{\mathrm{lu}}^{K_{\mathrm{lu}}};\, Z_{\mathrm{ch}}^1,\, Z_{\mathrm{ch}}^2, \ldots,\, Z_{\mathrm{ch}}^{K_{\mathrm{ch}}}\big)$$

$$= \big(a_{\mathrm{lu}}^1,\, a_{\mathrm{lu}}^2, \ldots,\, a_{\mathrm{lu}}^{K_{\mathrm{lu}}};\, |Z_{\mathrm{ch}}^1|\, e^{i\varphi_{\mathrm{ch}}^1},\, |Z_{\mathrm{ch}}^2|\, e^{i\varphi_{\mathrm{ch}}^2}, \ldots,\, |Z_{\mathrm{ch}}^{K_{\mathrm{ch}}}|\, e^{i\varphi_{\mathrm{ch}}^{K_{\mathrm{ch}}}}\big),$$

then the transformation

$$\mathbf{f}(\mathbf{z}) \to \mathrm{M}\cdot\mathbf{f}(\mathbf{z}) = \Big(\sum_{i=1}^{K_{\mathrm{lu}}} [a_{\mathrm{lu}}^i \cdot \mathbf{e}_{\mathrm{lu}}^i] + \sum_{j=1}^{K_{\mathrm{ch}}} [\,|Z_{\mathrm{ch}}^j|\, e^{i\varphi_{\mathrm{ch}}^j} \cdot \mathbf{E}_{\mathrm{ch}}^j]\Big) \cdot \Big(\sum_{i=1}^{K_{\mathrm{lu}}} [f_{\mathrm{lu}}^i(\mathbf{z}) \cdot \mathbf{e}_{\mathrm{lu}}^i] + \sum_{j=1}^{K_{\mathrm{ch}}} [\mathbf{f}_{\mathrm{ch}}^j(\mathbf{z}) \cdot \mathbf{E}_{\mathrm{ch}}^j]\Big)$$

$$= \Big(\sum_{i=1}^{K_{\mathrm{lu}}} [a_{\mathrm{lu}}^i f_{\mathrm{lu}}^i(\mathbf{z}) \cdot \mathbf{e}_{\mathrm{lu}}^i] + \sum_{j=1}^{K_{\mathrm{ch}}} [\,|Z_{\mathrm{ch}}^j|\, e^{i\varphi_{\mathrm{ch}}^j} \mathbf{f}_{\mathrm{ch}}^j(\mathbf{z}) \cdot \mathbf{E}_{\mathrm{ch}}^j]\Big)$$

changes the multiluminances, multihues, and multisaturations. The set of such transformations forms the group

$$\mathbf{MLCG}(\mathrm{A}_k(\mathbf{R}|1, \varepsilon^1, \varepsilon^2, \ldots, \varepsilon^{K-1}))$$

$$= \{(a_{\mathrm{lu}}^1, a_{\mathrm{lu}}^2, \ldots, a_{\mathrm{lu}}^{K_{\mathrm{lu}}};\, Z_{\mathrm{ch}}^1, Z_{\mathrm{ch}}^2, \ldots, Z_{\mathrm{ch}}^{K_{\mathrm{ch}}}) \,|\, (a_{\mathrm{lu}}^1, a_{\mathrm{lu}}^2, \ldots, a_{\mathrm{lu}}^{K_{\mathrm{lu}}} \in \mathbf{R}^+) \,\&\, (Z_{\mathrm{ch}}^1, Z_{\mathrm{ch}}^2, \ldots, Z_{\mathrm{ch}}^{K_{\mathrm{ch}}} \in \mathbf{C})\}.$$

We suppose that the brain can use hypercomplex algebras to mentally change the multiluminance and multichrominance of multichannel images that arise in the brain memory on the so-called "screen of mind," for example, during sleep.

4. SYNTHESIS OF COLOR TRANSFORMATIONS, WAVELETS, AND SPLINES

4.1. Orthounitary transformations of color images. The classical spectral analysis based on orthogonal and unitary transformations plays a pivotal role in digital image processing. Transformations similar to the discrete Fourier transform and discrete Walsh transform are extensively used in various applications (filtering, compression, spectral density estimation, and so on). The question naturally arises about the synthesis of transformations of color (triplet-valued) images. In this section, we propose a wide class of so-called orthounitary transformations for color image processing.

Discrete 2D color $(N\times N)$-images $\mathbf{f}_{\mathrm{col}} := \big[\mathbf{f}_{\mathrm{col}}(i,j)\big]_{i,j=1}^N$ are defined as 2D $(N\times N)$-arrays with pixels written in the (R,G,B) or LC formats. Here every color pixel $\mathbf{f}_{\mathrm{col}}(i,j)$ at a position (i,j) is a triplet number written in the (R,G,B) or LC formats, respectively. All the images $\mathbf{f}_{\mathrm{col}} := [\mathbf{f}_{\mathrm{col}}(i,j)]_{i,j=1}^N$ form the N^2-dimensional vector space $(\mathrm{Alg}_3^{\mathrm{Vis}})^{N^2}$ over a triplet algebra.

Definition 4. The quantity $\mathrm{N}_k(\mathbf{f}_{\mathrm{col}}) := \sum\limits_{(i,j)\in\mathbf{Z}_N^2} \|\mathbf{f}_{\mathrm{col}}(i,j)\|_k$ is called the *norm of the image* $\mathbf{f}_{\mathrm{col}}$, where $\|\cdot\|_k$, $k=1,2,3$, is one of the three triplet norms (3).

Definition 5. The linear operator $\mathrm{L}_{2\mathrm{D}}\colon (\mathrm{Alg}_3^{\mathrm{Vis}})^{N^2} \to (\mathrm{Alg}_3^{\mathrm{Vis}})^{N^2}$ is said to be *orthounitary* (or color) if it preserves the norm of a color image.

Note that orthogonal transformations preserve the norm of real-valued (gray) images and unitary transformations preserve the norm of complex-valued (bichromatic) images. Since a triplet number in the LC format is a kind of aggregate of real and complex numbers, it is natural to call these transformations orthounitary or color. Such transformations can be synthesized in the LC format using arbitrary orthogonal $O_{2\mathrm{D}}$ and unitary $U_{2\mathrm{D}}$ 2D transformations as follows:

$$\mathrm{L}_{2\mathrm{D}} = O_{2\mathrm{D}}\mathbf{e}_{\mathrm{lu}} + U_{2\mathrm{D}}\mathbf{E}_{\mathrm{ch}} = (O_{1\mathrm{D}} \otimes O_{1\mathrm{D}})\mathbf{e}_{\mathrm{lu}} + (U_{1\mathrm{D}} \otimes U_{1\mathrm{D}})\mathbf{E}_{\mathrm{ch}},$$

where $\otimes$ is the tensor product. Hence, each pair $(O_{1\mathrm{D}}, U_{1\mathrm{D}})$ of orthogonal $O_{1\mathrm{D}}$ and unitary $U_{1\mathrm{D}}$ transformations generates an orthounitary transformation $\mathrm{L}_{2\mathrm{D}} = (O_{1\mathrm{D}} \otimes O_{1\mathrm{D}})\mathbf{e}_{\mathrm{lu}} + (U_{1\mathrm{D}} \otimes U_{1\mathrm{D}})\mathbf{E}_{\mathrm{ch}}$.

Some color conversions are shown in Table 1 as an example.

Table 1

	F	$\dot{W}$	$\dot{H}d$	$\dot{W}v$
W	$W \cdot \mathbf{e}_{\mathrm{lu}} + F \cdot \mathbf{E}_{\mathrm{ch}}$	$W \cdot \mathbf{e}_{\mathrm{lu}} + \dot{W} \cdot \mathbf{E}_{\mathrm{ch}}$	$W \cdot \mathbf{e}_{\mathrm{lu}} + \dot{H}d \cdot \mathbf{E}_{\mathrm{ch}}$	$W \cdot \mathbf{e}_{\mathrm{lu}} + \dot{W}v \cdot \mathbf{E}_{\mathrm{ch}}$
Hd	$Hd \cdot \mathbf{e}_{\mathrm{lu}} + F \cdot \mathbf{E}_{\mathrm{ch}}$	$Hd \cdot \mathbf{e}_{\mathrm{lu}} + \dot{W} \cdot \mathbf{E}_{\mathrm{ch}}$	$Hd \cdot \mathbf{e}_{\mathrm{lu}} + \dot{H}d \cdot \mathbf{E}_{\mathrm{ch}}$	$Hd \cdot \mathbf{e}_{\mathrm{lu}} + \dot{W}v \cdot \mathbf{E}_{\mathrm{ch}}$
Ht	$Ht \cdot \mathbf{e}_{\mathrm{lu}} + F \cdot \mathbf{E}_{\mathrm{ch}}$	$Ht \cdot \mathbf{e}_{\mathrm{lu}} + \dot{W} \cdot \mathbf{E}_{\mathrm{ch}}$	$Ht \cdot \mathbf{e}_{\mathrm{lu}} + \dot{H}d \cdot \mathbf{E}_{\mathrm{ch}}$	$Ht \cdot \mathbf{e}_{\mathrm{lu}} + \dot{W}v \cdot \mathbf{E}_{\mathrm{ch}}$
Hr	$Hr \cdot \mathbf{e}_{\mathrm{lu}} + F \cdot \mathbf{E}_{\mathrm{ch}}$	$Hr \cdot \mathbf{e}_{\mathrm{lu}} + \dot{W} \cdot \mathbf{E}_{\mathrm{ch}}$	$Hr \cdot \mathbf{e}_{\mathrm{lu}} + \dot{H}d \cdot \mathbf{E}_{\mathrm{ch}}$	$Hr \cdot \mathbf{e}_{\mathrm{lu}} + \dot{W}v \cdot \mathbf{E}_{\mathrm{ch}}$
Wv	$Wv \cdot \mathbf{e}_{\mathrm{lu}} + F \cdot \mathbf{E}_{\mathrm{ch}}$	$Wv \cdot \mathbf{e}_{\mathrm{lu}} + \dot{W} \cdot \mathbf{E}_{\mathrm{ch}}$	$Wv \cdot \mathbf{e}_{\mathrm{lu}} + \dot{H}d \cdot \mathbf{E}_{\mathrm{ch}}$	$Wv \cdot \mathbf{e}_{\mathrm{lu}} + \dot{W}v \cdot \mathbf{E}_{\mathrm{ch}}$

In Table 1 $O_{1\mathrm{D}} = W$, Hd, Ht, Hr, Wv are orthogonal Walsh, Hadamard, Hartley, Haar transforms, and a wavelet transform, respectively; $U_{1\mathrm{D}} = F$, $\dot{W}$, $\dot{H}d$, $\dot{W}v$ are a unitary Fourier transform, complex-valued Walsh and Hadamard transforms, and, finally, a complex-valued wavelet transform, respectively.

If $O = \big[\varphi_k(n)\big]_{k,n=0}^{N-1}$ and $U = \big[\psi_k(n)\big]_{k,n=0}^{N-1}$ are general orthogonal and unitary transformations whose rows constitute bases of real-valued and complex-valued functions $\big\{\varphi_k(n)\big\}_{k,n=0}^{N-1}$ and $\big\{\psi_k(n)\big\}_{k,n=0}^{N-1}$, then the expression

$$\begin{aligned}(O \otimes O)\mathbf{e}_{\mathrm{lu}} + (U \otimes U)\mathbf{E}_{\mathrm{ch}} &= ([\varphi_{k_1}(n_1)] \otimes [\varphi_{k_2}(n_2)])\mathbf{e}_{\mathrm{lu}} + [\psi_{k_1}(n_1)] \otimes [\psi_{k_2}(n_2)]\mathbf{E}_{\mathrm{ch}} \\ &= [\varphi_{k_1}(n_1)\,\varphi_{k_2}(n_2)]\mathbf{e}_{\mathrm{lu}} + [\psi_{k_1}(n_1)\,\psi_{k_2}(n_2)]\mathbf{E}_{\mathrm{ch}}\end{aligned}$$

represents a color transformation, where

$$\big\{\varphi_{k_1}(n_1)\,\varphi_{k_2}(n_2)\big\}_{k_1,k_2=0,\,n_1,n_2=0}^{N-1,\,N-1} \quad \text{and} \quad \big\{\psi_{k_1}(n_1)\,\psi_{k_2}(n_2)\big\}_{k_1,k_2=0,\,n_1,n_2=0}^{N-1,\,N-1}$$

form N^2 orthogonal and unitary basis functions.

4.2. Orthounitary (color) wavelets. Let $\psi^R(x)$ be a real-valued mother wavelet, and let $\psi^R_{s,\tau}(x)$ be its shifted and scaled versions:

$$\psi^R_{s,\tau}(x) = \big(\sqrt{|s|}\,\big)^{-1}\psi^R\Big(\frac{x-\tau}{s}\Big), \quad s, \tau \in R, \quad s \neq 0.$$

They form an orthogonal basis of the space $L_2(\mathbf{R})$. We construct color wavelets as combinations of luminance and chromatic components. As the first component, we use the real-valued wavelet $\psi^R(x)$. The chromatic component is defined as an analytic complex-valued signal of the form

$$\psi^{\mathrm{Ch}}_{s,\tau}(x) = \psi^R_{s,\tau}(x) + j\,\mathrm{H}_1\{\psi^R_{s,\tau}(x)\},$$

where $\mathrm{H}_1\{\cdot\}$ is the scalar Hilbert transform. Using idempotents $\mathbf{e}_{\mathrm{lu}}$ and $\mathbf{E}_{\mathrm{Ch}}$, we construct the color basis wavelets

$$\begin{aligned}\psi^{\mathrm{Col}}_{s,\tau}(x) &= \varphi^{\mathrm{lu}}_{s,\tau}(x)\cdot\mathbf{e}_{\mathrm{lu}} + \psi^{\mathrm{Ch}}_{s,\tau}(x)\cdot\mathbf{E}_{\mathrm{Ch}} = \varphi^{\mathrm{lu}}_{s,\tau}(x)\cdot\mathbf{e}_{\mathrm{lu}} + [\varphi^{\mathrm{lu}}_{s,\tau}(x) + j\,\mathrm{H}\{\varphi^{\mathrm{lu}}_{s,\tau}(x)\}]\cdot\mathbf{E}_{\mathrm{Ch}}\\ &= \varphi^{\mathrm{lu}}_{s,\tau}(x)\cdot[\mathbf{e}_{\mathrm{lu}} + \mathbf{E}_{\mathrm{Ch}}] + j\,\mathrm{H}\{\varphi^{\mathrm{lu}}_{s,\tau}(x)\}\cdot\mathbf{E}_{\mathrm{Ch}} = \varphi^{\mathrm{lu}}_{s,\tau}(x)\cdot\mathbf{I}_3 + j\,\mathrm{H}\{\varphi^{\mathrm{lu}}_{s,\tau}(x)\}\cdot\mathbf{E}_{\mathrm{Ch}},\end{aligned}$$

where $\varphi^{\mathrm{lu}}_{s,\tau}(x) = \varphi^{R}_{s,\tau}(x)$ is the real-valued (luminance) component of the wavelet, $\psi^{\mathrm{Ch}}_{s,\tau}(x)$ is its complex-valued (chromatic) component, and $\mathbf{I}_3 = \mathbf{e}_{\mathrm{lu}} + \mathbf{E}_{\mathrm{Ch}}$ is the identity (3×3)-transformation. Obviously, the analytic functions of two variables

$$\begin{aligned}\psi^{\mathrm{Col}}_{s_1,\tau_1}(x)\,\psi^{\mathrm{Col}}_{s_2,\tau_2}(y) &= \varphi^{\mathrm{lu}}_{s_1,\tau_1}(x)\,\varphi^{\mathrm{lu}}_{s_2,\tau_2}(y)\cdot\mathbf{e}_{\mathrm{lu}} + [\varphi^{\mathrm{lu}}_{s_1,\tau_1}(x)\,\varphi^{\mathrm{lu}}_{s_2,\tau_2}(y) + j\,\mathrm{H}_2\{\varphi^{\mathrm{lu}}_{s_1,\tau_1}(x)\,\varphi^{\mathrm{lu}}_{s_2,\tau_2}(y)\}]\cdot\mathbf{E}_{\mathrm{Ch}}\\ &= \varphi^{\mathrm{lu}}_{s_1,\tau_1}(x)\,\varphi^{\mathrm{lu}}_{s_2,\tau_2}(y)\cdot\mathbf{I}_3 + j\,\mathrm{H}_2\{\varphi^{\mathrm{lu}}_{s_1,\tau_1}(x)\,\varphi^{\mathrm{lu}}_{s_2,\tau_2}(y)\}\cdot\mathbf{E}_{\mathrm{Ch}}\end{aligned}$$

form two-dimensional wavelets, where $\mathrm{H}_2\{\cdot\}$ is the two-dimensional Hilbert transform.

We define a two-dimensional color (orthounitary) wavelet transform as

$$F^{\mathrm{col}}_{\mathrm{COUT}}(s_1,\tau_1,s_2,\tau_2) = \big(\sqrt{|s_1|\,|s_2|}\,\big)^{-1}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\mathbf{f}_{\mathrm{col}}(x,y)\,\psi^{\mathrm{Col}}_{s_1,\tau_1}(x)\,\psi^{\mathrm{Col}}_{s_2,\tau_2}(y)\,dx\,dy.$$

4.3. Orthounitary splines. Color splines can be constructed similarly to color wavelets. Let $\mathrm{Spl}(x)$ be a real-valued spline. A color spline is the triplet-valued function

$$\begin{aligned}\mathrm{Spl}^{\mathrm{Col}}(x) &= \mathrm{Spl}(x)\cdot\mathbf{e}_{\mathrm{lu}} + [\mathrm{Spl}(x) + j\,\mathrm{H}\{\mathrm{Spl}(x)\}]\cdot\mathbf{E}_{\mathrm{Ch}}\\ &= \mathrm{Spl}(x)\cdot[\mathbf{e}_{\mathrm{lu}} + \mathbf{E}_{\mathrm{Ch}}] + j\,\mathrm{H}\{\mathrm{Spl}(x)\}\cdot\mathbf{E}_{\mathrm{Ch}} = \mathrm{Spl}(x)\cdot\mathbf{I}_3 + j\,\mathrm{H}\{\mathrm{Sp}(x)\}\cdot\mathbf{E}_{\mathrm{Ch}}.\end{aligned}\tag{4}$$

Suppose, for example, that $\mathrm{Spl}(x)\equiv B\mathrm{Spl}(x)$ is an arbitrary B-spline, which is constructed using an iterative convolution of a rectangular pulse:

$$B\mathrm{Spl}_0(x) = \begin{cases}1, & -1/2 < x < 1/2,\\ 1/2, & |x| = 1/2,\\ 0 & \text{otherwise},\end{cases}\qquad B\mathrm{Spl}_n(x) = (B\mathrm{Spl}_{n-1} * B\mathrm{Spl}_0)(x),$$

where $*$ is the convolution symbol. In accordance with (4), the color scalar B-spline has the form

$$B\mathrm{Spl}^{\mathrm{Col}}_n(x) = B\mathrm{Spl}_n(x)\cdot\mathbf{I}_3 + j\,\mathrm{H}\{B\mathrm{Spl}_n(x)\}\cdot\mathbf{E}_{\mathrm{Ch}}.$$

Obviously, the functions

$$B\mathrm{Spl}^{\mathrm{Col}}_n(x)\cdot B\mathrm{Spl}^{\mathrm{Col}}_n(y) = B\mathrm{Spl}_n(x)\cdot B\mathrm{Spl}_n(y)\cdot\mathbf{I}_3 + j\,\mathrm{H}_2\{B\mathrm{Spl}_n(x)\cdot B\mathrm{Spl}_n(y)\}\cdot\mathbf{E}_{\mathrm{Ch}}$$

are two-dimensional splines.

CONCLUSIONS

A new algebraic approach to mathematical models of multichannel images based on commutative hypercomplex algebras is developed. The aim of the paper has been to show that hypercomplex algebras are an adequate mathematical tool for describing multichannel images. A sufficient number of arguments can be given in favor of the fact that the brain of animals gained the evolved ability to operate with hypercomplex numbers in the process of image processing and recognition. Therefore, the brain of animals can be considered as a computer operating in a certain hypercomplex algebra.

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