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FAST FRACTIONAL FOURIER TRANSFORMS BASED ON INFINITESIMAL FOURIER TRANSFORM




Introduction

The idea of fractional powers of the Fourier operator $\{\mathcal{F}^a\}_{a=0}^4$ appears in the mathematical literature (Wiener, 1929; Condon, 1937; Kober, 1939; Bargmann, 1961; Mitra, 2001). This idea is to consider the eigen-value decomposition of the Fourier transform $\mathcal{F}$ in terms of eigen-values $\lambda_n = e^{jn\pi/2}$ and eigen-functions in the form of the Hermite functions. The family of FrFT $\{\mathcal{F}^a\}_{a=0}^4$ is constructed by replacing the n-th eigen-value $\lambda_n = e^{jn\pi/2}$ by its a-th power $\lambda_n^a = e^{jn\pi a/2}$, for a between 0 and 4. This value is called the transform order. There is angle parameterization $\{\mathcal{F}^\alpha\}_{\alpha=0}^{2\pi}$ where $\alpha = \pi a/2$ is a new angle parameter. Since this family depends on a single parameter, the fractional operators $\{\mathcal{F}^a\}_{a=0}^4$ (or $\{\mathcal{F}^\alpha\}_{\alpha=0}^{2\pi}$) form the Fourier-Hermite one-parameter strongly continuous unitary multiplicative group $\mathcal{F}^a\mathcal{F}^b = \mathcal{F}^{a\oplus b}$ (or $\mathcal{F}^\alpha\mathcal{F}^\beta = \mathcal{F}^{\alpha\oplus\beta}$), where $a\oplus b = (a+b) \bmod 4$ (or $\alpha\oplus\beta = (\alpha+\beta) \bmod 2\pi$) and $\mathcal{F}^0 = I$. The identical and classical Fourier transformations are both the special cases of the FrFTs. They correspond to $\alpha = 0$ ($\mathcal{F}^0 = I$) and $\alpha = \pi/2$ ($\mathcal{F}^{\pi/2} = \mathcal{F}$), respectively.

In 1980, Namias reinvented the fractional Fourier transform (FrFT) again in his paper (Namias, 1980). He used the FrFT in the context of quantum mechanics as a way to solve certain problems involving quantum harmonic oscillators. He not only stated the standard definition for the FrFT, but also developed an operational calculus for this new transform. This approach was extended by McBride and Kerr (1987). Then Mendlovic and Ozaktas (1993) introduced the FrFT into the field of optics. Afterwards, Lohmann (1993) reinvented the FrFT based on the Wigner–distribution function and opened the FrFT to bulk–optics applications. It has been rediscovered signal and image processing (Almeida, 1994). In these cases FrFT allows us to extract time-frequency information from the signal. A recent state of the art can be found by H. Ozaktas et al. (2001). In the series of papers (Labunets E. & Labunets V., 1998; Rundblad et al., 1999 *a*, *b*; Rundblad–Labunets et al., 1999; Creutzburg et al., 1999; Ozaktas et al., 2001), we developed a wide class of classical and quantum fractional transforms. In this paper, are introduced the infinitesimal Fourier transform and discussed the relationship of the fractional Fourier transform to the Schrödinger operator of the quantum harmonic oscillator. Up to now, the fractional Fourier spectra $F^{\alpha_i} = \mathcal{F}^{\alpha_i}\{f\}$, $i = 1, 2, ..., M$, has been digitally computed using classical approach based on the FFT. This method maps the N samples of the original function f to the NM samples of the set of spectra $\left\{F^{\alpha_i}\right\}_{i=1}^{M}$, which requires $MN\left(2+\log_2 N\right)$ multiplications and $MN\log_2 N$ additions. This paper develops a new numerical algorithm, which requires $2MN$ multiplications and $3MN$ additions and which is based on the infinitesimal Fourier transform.

Eigen-decomposition and fractional discrete transforms

The eigen–decomposition (ED) is a tool of both practical and theoretical importance in digital signal and image processing. The ED transforms are defined following way. Let $\mathrm{F}=\left[F_k(i)\right]_{k,i=0}^{N-1}$ be an arbitrary discrete symmetric $(N\times N)$–transform, λ_n and $\Psi_n(t)$, $n=0,1,\ldots,N-1,$ be its eigen–values and eigen–vectors, respectively. Let $\mathbf{U}=\left[\Psi_0(i)\mid\Psi_1(i)\mid\vdots\mid\Psi_{N-1}(i)\right]$ be the matrix of eigen–vectors of the F–transform. Then $\mathbf{U}^{-1}\mathrm{F}\,\mathbf{U}=\mathbf{Diag}\{\lambda_n\}$. Hence, we have the following eigen–decomposition: $\mathrm{F}=\left[F_k(i)\right]:=\mathbf{U}\Lambda\mathbf{U}^{-1}$.

Definition 1 [12--16]. *For an arbitrary real numbers* $a_0,\ldots,a_{N-1}$ *we introduce the multi–parametric* F *–transform*

$$\mathrm{F}^{(a_0,\ldots,a_{N-1})}:=\mathbf{U}\left\{\mathrm{diag}\left(\lambda_0^{a_0},\ldots,\lambda_{N-1}^{a_{N-1}}\right)\right\}\mathbf{U}^{-1}. \tag{51}$$

If $a_0=\ldots=a_{N-1}\equiv a$ then this transform is called fractional F –transform. For this transform we have

$$\mathrm{F}^{a}:=\mathbf{U}\left\{\mathrm{diag}\left(\lambda_0^{a},\ldots,\lambda_{N-1}^{a}\right)\right\}\mathbf{U}^{-1}=\mathbf{U}\Lambda^{a}\mathbf{U}^{-1}. \tag{52}$$

The zeroth-order fractional F –transform is equal to the identity transform: $\mathrm{F}^{0}=\mathbf{U}\Lambda^{0}\mathbf{U}^{-1}=\mathbf{U}\mathbf{U}^{-1}=\mathbf{I}$ and the first-order fractional Fourier transform operator $\mathrm{F}^{1}=\mathrm{F}$ is equal to the initial F –transform $\mathrm{F}^{1}=\mathbf{U}\Lambda\mathbf{U}^{-1}$.

The families $\left\{\mathrm{F}^{(a_0,\ldots,a_{N-1})}\right\}_{(a_0,\ldots,a_{N-1})\in\mathbf{R}^N}$ and $\left\{\mathrm{F}^{a}\right\}_{a\in\mathbf{R}}$ form multi- and one-parameter continuous unitary groups with multiplications $\mathrm{F}^{(a_0,\ldots,a_{N-1})}\mathrm{F}^{(b_0,\ldots,b_{N-1})}=$ $=\mathrm{F}^{(a_0+b_0,\ldots,a_{N-1}+b_{N-1})}$ and $\mathrm{F}^{a}\mathrm{F}^{b}=\mathrm{F}^{a+b}$. Indeed, $\mathrm{F}^{a}\mathrm{F}^{b}=\mathbf{U}\Lambda^{a}\mathbf{U}^{-1}\cdot\mathbf{U}\Lambda^{b}\mathbf{U}^{-1}=\mathbf{U}\Lambda^{a+b}\mathbf{U}^{-1}=\mathrm{F}^{a+b}$ and

$$\mathrm{F}^{(a_0,\ldots,a_{N-1})}\mathrm{F}^{(b_0,\ldots,b_{N-1})}=$$
$$\mathbf{U}\left\{\mathrm{diag}\left(\lambda_0^{a_0},\ldots,\lambda_{N-1}^{a_{N-1}}\right)\right\}\mathbf{U}^{-1}\cdot\mathbf{U}\left\{\mathrm{diag}\left(\lambda_0^{b_0},\ldots,\lambda_{N-1}^{b_{N-1}}\right)\right\}\mathbf{U}^{-1}=$$
$$=\mathbf{U}\left\{\mathrm{diag}\left(\lambda_0^{a_0+b_0},\ldots,\lambda_{N-1}^{a_{N-1}+b_{N-1}}\right)\right\}\mathbf{U}^{-1}=\mathrm{F}^{(a_0+b_0,\ldots,a_{N-1}+b_{N-1})}.$$

Let $\mathrm{F}=\left[F_k(i)\right]_{k,i=0}^{N-1}$ be discrete Fourier $(N\times N)$–transform (DFT), then $\lambda_n=e^{j\pi n/2}\in\{\pm1,\pm j\}$ and $\left\{\Psi_n(t)\right\}_{n=0}^{N-1}$ are the Kravchuk polynomials. Integer values of a for F^{a} simply correspond to repeated application of the ordinary DFT and negative integer values correspond to repeated application of the inverse DFT. For instance, $\mathrm{F}^{2}=\mathrm{F}\mathrm{F}$ corresponds to the Fourier transform of the Fourier transform $\mathrm{F}^{2}f=\mathrm{F}(\mathrm{F}f)$ and $\mathrm{F}^{-2}=\mathrm{F}^{-1}\mathrm{F}^{-1}$ corresponds to the inverse Fourier transform of the inverse Fourier transformn $\mathrm{F}^{-2}f=\mathrm{F}^{-1}\left(\mathrm{F}^{-1}f\right)$.

Definition 2. The multi–parametric and fractional DFT are

$$\mathrm{F}^{(a_0,\ldots,a_{N-1})}:=\mathbf{U}\left\{\mathrm{diag}\left(e^{j\pi 0a_0/2},e^{j\pi 1a_1/2}\ldots,e^{j\pi(N-1)a_{N-1}/2}\right)\right\}\mathbf{U}^{-1},$$
$$\mathrm{F}^{a}:=\mathbf{U}\left\{\mathrm{diag}\left(e^{j\pi na/2}\right)\right\}\mathbf{U}^{-1}$$

and

$$\mathrm{F}^{(\alpha_0,\ldots,\alpha_{N-1})}:=\mathbf{U}\left\{\mathrm{diag}\left(e^{j0\alpha_0},e^{j1\alpha_1}\ldots,e^{j(N-1)\alpha_{N-1}}\right)\right\}\mathbf{U}^{-1},\quad \mathrm{F}^{\alpha}:=\mathbf{U}\left\{\mathrm{diag}\left(e^{jn\alpha}\right)\right\}\mathbf{U}^{-1}$$

in $a-$ and $\alpha-$parameterizations, respectively.

The parameters $(a_0,\dots,a_{N-1})$ and a can be any real value. However, the operators $\mathrm{F}^{(a_0,\dots,a_{N-1})}$ and F^{a} are periodic in each parameter with period 4 since $\mathrm{F}^{4}=I$ and hence $\mathrm{F}^{(a_0,\dots,a_{N-1})}\mathrm{F}^{(b_0,\dots,b_{N-1})}=\mathrm{F}^{\left(a_0\underset{4}{\oplus}b_0,\dots,a_{N-1}\underset{4}{\oplus}b_{N-1}\right)}$ and $\mathrm{F}^{a}\mathrm{F}^{b}=\mathrm{F}^{a\underset{4}{\oplus}b}$, where $a_i\underset{4}{\oplus}b_i=\left(a_i+b_i\right)\mathrm{mod}\,4,\ \forall i=0,1,\dots,N-1$.

Consequently, the ranges of $(a_0,\dots,a_{N-1})$ and a are $(\mathrm{Z}/4\mathrm{Z})^N=[0,4]^N$ (or $[-2,2]^N$) and $\mathrm{Z}/4\mathrm{Z}\equiv[0,4]$ (or $[-2,2]$).

In the case of α – parameterization we have $\alpha_i\underset{2\pi}{\oplus}\beta_i=\left(\alpha_i+\beta_i\right)\mathrm{mod}\,2\pi,\ \forall i=0,1,\dots,N-1$. Consequently, the ranges of $(\alpha_0,\dots,\alpha_{N-1})$ and α are $(\mathrm{Z}/2\pi\mathrm{Z})^N=[0,2\pi)^N$ (or $[-\pi,\pi)^N$) and $\mathrm{Z}/2\pi\mathrm{Z}\equiv[0,2\pi)$ (or $[-\pi,\pi)$), respectively.

Canonical FrFT

The continuous Fourier transform is a unitary operator F that maps square-integrable functions on square-integrable ones, and is represented on these functions $f(x)$ by the well-known integral:

$$\left(\mathrm{F}f\right)(y)=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}f(x)e^{-jyx}dx. \qquad (53)$$

Relevant properties are that the square $\left(\mathrm{F}^2 f\right)(x)=f(-x)$ is the inversion operator, and that its fourth power, $\left(\mathrm{F}^4 f\right)(x)=f(x)$, is the identity; hence $\mathrm{F}^3=\mathrm{F}^{-1}$.. The operator F thus generates a cyclic group of order 4. Bargmann (1961) extended the Fourier transform and he gave definition of the FrFT, one based on Hermite polynomials as an integral transformation. If $H_n(x)$ is a Hermite polynomial of order n, where $H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{x^2}$, then for $n\in\mathrm{N}_0:=\{0,1,2,\dots\}$, functions $\Psi_n(x)=\frac{1}{\sqrt{2^n n!\sqrt{\pi}}}H_n(x)e^{-x^2/2}$ are eigen–functions of the Fourier transform

$$\mathrm{F}\left[\Psi_n(x)\right]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Psi_n(x)e^{2\pi jyx}dx=\lambda_n\Psi_n(y)=e^{-j\frac{\pi}{2}n}\Psi_n(y)$$

with $\lambda_n=j^n=e^{-j\frac{\pi}{2}n}$ being the eigen–value corresponding to the nth eigen–function. According to Bargmann the fractional Fourier transform $\mathrm{F}^{\alpha}=\left[K^{\alpha}(x,y)\right]$ is defined through its the eigen-functions as

$$K^{\alpha}(x,y):=U\left\{diag\left(e^{-j\alpha n}\right)\right\}U^{-1}=\sum_{n=0}^{\infty}e^{-j\alpha n}\Psi_n(x)\Psi_n(y). \qquad (54)$$

Hence,

$$K^{\alpha}(x,y):=\sum_{n=0}^{\infty}e^{-j\alpha n}\Psi_n(x)\Psi_n(y)=e^{-\left(x^2+y^2\right)}\sum_{n=0}^{\infty}\frac{e^{-j\alpha n}H_n(x)H_n(y)}{2^n n!\sqrt{\pi}}=$$
$$=\frac{1}{\sqrt{\pi}\sqrt{1-e^{-2j\alpha}}}\cdot\exp\left\{\frac{2xye^{-j\alpha}-e^{-2j\alpha}\left(x^2+y^2\right)}{1-e^{-2j\alpha}}\right\}\exp\left\{-\frac{\left(x^2+y^2\right)}{2}\right\}, \qquad (55)$$

where $F^{\alpha}(x,y)$ is the kernel of the FrFT. In the last step is used the Mehler formula (Goldstein, 1985)

$$\sum_{n=0}^{\infty}\frac{e^{-j\alpha n}H_n(x)H_n(y)}{2^n n!\sqrt{\pi}}=\frac{1}{\sqrt{\pi}\sqrt{1-e^{-2j\alpha}}}\exp\left\{\frac{2xye^{-j\alpha}-e^{-2j\alpha}\left(x^2+y^2\right)}{1-e^{-2j\alpha}}\right\}.$$

Expression (5) can be rewritten as

$$K^{\alpha}(x,y)=\sqrt{\frac{1-j\cot\alpha}{2\pi}}\exp\left\{\frac{j}{2\sin\alpha}\left[\left(x^2+y^2\right)\cos\alpha-2xy\right]\right\},$$

where $\alpha \neq \pi\mathbf{Z}$ (or $\alpha \neq 2\mathbf{Z}$). Obviously, a functions $\Psi_n(x)$ are eigen–functions of the fractional Fourier transform $\mathrm{F}^{\alpha}\left[\Psi_n(x)\right] = e^{jn\alpha}\Psi_n(x)$ corresponding to the *n*-th eigen–values $e^{jn\alpha}, n = 0,1,2,\ldots$ The FrFT is F^{α} a unitary operator that maps square-integrable functions $f(x)$ on square--integrable ones

$$F^{\alpha}(y) = \left(\mathrm{F}^{\alpha} f\right)(y) = \int_{x\in\mathbf{R}} f(x) K^{\alpha}(x,y)\,dx =$$

$$= \frac{e^{-\frac{j}{2}\left(\frac{\pi}{2}\hat{\alpha}-\alpha\right)}}{\sqrt{2\pi|\sin\alpha|}} \int_{x\in\mathbf{R}} f(x)\exp\left\{\frac{j}{2\sin\alpha}\left[\left(x^2+y^2\right)\cos\alpha - 2xy\right]\right\}dx.$$

There exist several algorithms for fast calculation of spectrum of the fractional Fourier transform $F^{\alpha}(y)$. But all of them are based on the following transform of the FrFT:

$$F^{\alpha}(y) = \left(\mathrm{F}^{\alpha} f\right)(y) = \frac{e^{-\frac{j}{2}\left(\frac{\pi}{2}\hat{\alpha}-\alpha\right)} e^{jy^2\frac{\cos\alpha}{2\sin\alpha}}}{\sqrt{2\pi|\sin\alpha|}} \int_{\mathbf{R}} \left[f(x) e^{j\frac{x^2}{2}\cot\alpha}\right] e^{-jxy}\,dx =$$

$$= A_{\alpha}(y)\cdot \mathrm{F}\left\{f(x)\cdot B_{\alpha}(x)\right\}(y),$$

where $A_{\alpha}(y) = \dfrac{e^{-\frac{j}{2}\left(\frac{\pi}{2}\hat{\alpha}-\alpha\right)} e^{jy^2\frac{\cos\alpha}{2\sin\alpha}}}{\sqrt{2\pi|\sin\alpha|}}$, $B_{\alpha}(x) = e^{j\frac{x^2}{2}\cot\alpha}$.

Let us introduce the uniformly discretization (or sampling) of the angle parameter α on M discrete values $\{\alpha_0, \alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_{M-1}\}$, where $\alpha_{i+1} = \alpha_i + \Delta\alpha$, $\alpha_i = i\Delta\alpha$ and $\Delta\alpha = 2\pi/M$. The set of M spectra $\left\{F^{\alpha_0}(y), F^{\alpha_1}(y), \ldots, F^{\alpha_{M-1}}(y)\right\}$ can be computed by applying the following sequence of steps for all $\{\alpha_0, \alpha_1, \ldots, \alpha_{M-1}\}$:

1. Compute products $f(x)B_{\alpha_k}(x)$, which require N multiplications.
2. Compute the Fast Fourier Transform (FFT) of $N\log_2 N$ multiplications and additions.
2. Multiply the result by $A_{\alpha}(y)$ (N multiplications).

This numerical algorithm requires $MN(2+\log_2 N)$ multiplications and $MN\log_2 N$ additions.

Infinitesimal Fourier Transform

In order to construct fast multi–parametric F –transform and fractional Fourier transform algorithms, we turn our attention to notion of a semigroup and its generator (infinitesimal operator). Let $\mathrm{L}_2(\mathbf{R},\mathbf{C})$ be a space of complex-valued functions (signals) and let $\mathbf{Op}(\mathrm{L}_2)$ be the Banach algebra of all bounded linear operators on $\mathrm{L}_2(\mathbf{R},\mathbf{C})$ endowed with the operator norm. A family $\{\mathbf{U}(\alpha)\}_{\alpha\in\mathbf{R}} \subset \mathbf{Op}(\mathrm{L}_2)$ is called a Hermite group on $\mathrm{L}_2(\mathbf{R},\mathbf{C})$ if it satisfies Abel's functional equation:

$$\mathbf{U}(\alpha+\beta) = \mathbf{U}(\alpha)\mathbf{U}(\beta) \text{ for all } \alpha,\beta\in\mathbf{R},$$
$$\mathbf{U}(0) = I$$

and the orbit maps $\alpha \to F^{\alpha} = \mathbf{U}(\alpha)\{f\}$ are continuous from $\mathbf{R}$ into $\mathrm{L}_2(\mathbf{R},\mathbf{C})$ for every $f \in \mathrm{L}_2(\mathbf{R},\mathbf{C})$.

Definition 3 (Goldstein, 1985). The infinitesimal generator $\mathbf{A}(0)$ of the group $\{\mathbf{U}(\alpha)\}_{\alpha\in\mathbf{R}}$ and infinitesimal transform $\mathbf{U}(d\alpha)$ are defined as follows:

$$\mathbf{A}(0) = \left.\frac{\partial\mathbf{U}(\alpha)}{\partial\alpha}\right|_{\alpha=0}, \quad \mathbf{U}(d\alpha) = I + d\mathbf{U}(0) = I + \mathbf{A}(0)\,d\alpha.$$

Obviously,

$$\mathbf{U}(\alpha_0+d\alpha)=\mathbf{U}(\alpha_0)+d\mathbf{U}(\alpha_0)=\mathbf{U}(\alpha_0)+\left.\frac{\partial\mathbf{U}(\alpha)}{\partial\alpha}\right|_{\alpha_0}d\alpha=$$
$$=\mathbf{U}(\alpha_0)+\mathbf{A}(\alpha_0)d\alpha.$$

But

$$\mathbf{U}(\alpha_0+d\alpha)=\mathbf{U}(d\alpha_0)\mathbf{U}(\alpha_0)=\left[\mathbf{I}+d\mathbf{U}(0)\right]\mathbf{U}(\alpha_0)=\mathbf{U}(\alpha_0)+\left.\frac{\partial\mathbf{U}(\alpha)}{\partial\alpha}\right|_{0}\mathbf{U}(\alpha_0)d\alpha=$$
$$=\mathbf{U}(\alpha_0)+\mathbf{A}(0)\mathbf{U}(\alpha_0)d\alpha=\left[\mathbf{I}+\mathbf{A}(0)\right]\mathbf{U}(\alpha_0)d\alpha.$$

Hence, $\mathbf{A}(\alpha_0)=\mathbf{A}(0)\mathbf{U}(\alpha_0)$ and

$$F^{\alpha_0+d\alpha}(y)=\left(\mathbf{U}(\alpha_0+d\alpha)F\right)(y)=\left(\left[\mathbf{I}+\mathbf{A}(0)\right]\mathbf{U}(\alpha_0)F\right)(y)d\alpha=$$
$$=\left[\mathbf{I}+\mathbf{A}(0)\right]F^{\alpha_0}(y)d\alpha.$$

Define now the linear operator $\mathrm{H}=\frac{1}{2}\left(\frac{d^2}{dx^2}-x^2+1\right)$. It is known that

$$\mathrm{H}\Psi_n(x)=\frac{1}{2}\left(\frac{d^2}{dx^2}-x^2+1\right)\Psi_n(x)=n\Psi_n(x) \qquad (56)$$

From (4) and (6) we have

$$j\frac{\partial}{\partial\alpha}F^{\alpha}(y)\Big|_{\alpha=0}=j\frac{\partial}{\partial\alpha}\left\{\mathrm{F}^{\alpha}F\right\}(y)\Big|_{\alpha=0}=\sum_{n=0}^{\infty}n\Psi_n(y)\int\Psi_n(x)f(x)dx,$$
$$\mathrm{H}F^{\alpha}(y)=\left\{\sum_{n=0}^{\infty}n\Psi_n(y)\int\Psi_n(x)f(x)dx\right\}.$$

Therefore $j\frac{\partial F^{\alpha}(y)}{\partial\alpha}=\mathrm{H}F^{\alpha}(y)$, $\frac{\partial F^{\alpha}(y)}{F^{\alpha}(y)}=-j\mathrm{H}\partial\alpha.$

The solution of this equation is given by

$$F^{\alpha}(y)=\left\{e^{-j\alpha\mathrm{H}}F\right\}(y)\ \text{ and }\ \mathrm{F}^{\alpha}=e^{-j\alpha\mathrm{H}}=e^{-j\alpha\left[\frac{1}{2}\left(\frac{d^2}{dy^2}-y^2+1\right)\right]}.$$

Obviously,

$$\mathrm{F}^{\alpha+d\alpha}=\mathrm{F}^{d\alpha}\mathrm{F}^{\alpha}\,\square\,\left(I+d\mathrm{F}^{\alpha}\right)\exp\left[-j\alpha\mathrm{H}\right]=$$
$$=\left(I+\frac{\partial\mathrm{F}^{\alpha}}{\partial\alpha}d\alpha\right)\exp\left(-j\alpha\mathrm{H}\right)=\left(I-j\mathrm{H}d\alpha\right)\exp\left(-j\alpha\mathrm{H}\right),$$

where the operator

$$\mathrm{F}^{d\alpha}=\left(I-j\mathrm{H}d\alpha\right)=\mathbf{I}-j\frac{1}{2}\left(\frac{d^2}{dx^2}-x^2+1\right)d\alpha \qquad (57)$$

is called the *infinitesimal Fourier transform* or the generator of the fractional Fourier transforms (Griffths, 2002).

Let us introduce the multiplication operators $\left(M_x f\right)(x)=xf(x)$ and $\left(M_y F\right)(y)=yF(y)$. Using the Fourier transform (3), the first of ones may be written as $M_x=\mathrm{F}\left(\frac{d}{jdy}\right)\mathrm{F}^{-1}$. Obviously,

$x^2=M_x^2=-\mathrm{F}\left(\frac{d^2}{dy^2}\right)\mathrm{F}^{-1}$. Then

$$\mathrm{F}^{d\alpha}=\mathbf{I}-j\frac{1}{2}\left(\frac{d^2}{dx^2}+\mathrm{F}\left(\frac{d^2}{dy^2}\right)\mathrm{F}^{-1}+1\right)d\alpha.$$

Discretization of x-domain with the interval discretization Δx is equal to the periodization of y-domain with the period periodization $2\pi/\Delta x$

$$\frac{d^2}{dx^2}+\mathrm{F}\left(\frac{d^2}{dy^2}\right)\mathrm{F}^{-1}+1\to D_{\Delta x}\left[\frac{d^2}{dx^2}\right]+\mathrm{F}\left(P_{2\pi/\Delta x}\left[\frac{d^2}{dy^2}\right]\right)\mathrm{F}^{-1}+1.$$

Discretization of y-domain with the interval discretization Δy is equal to the periodization of x-domain with the period periodization $2\pi/\Delta y$

$$P_{2\pi/\Delta y}D_{\Delta x}\left[\frac{d^2}{dx^2}\right]+\mathrm{F}\left(P_{2\pi/\Delta x}D_{\Delta y}\left[\frac{d^2}{dy^2}\right]\right)\mathrm{F}^{-1}+1.$$

An approximation for the second derivative can be given by the second order central difference operator

$$P_{2\pi/\Delta y}D_{\Delta x}\left(\frac{d^2}{dx^2}f\right)(n)\approx\left(f(n\underset{N}{\$}1)-2f(n)+f(n\underset{N}{\oplus}1)\right),$$

$$P_{2\pi/\Delta x}D_{\Delta y}\left(\frac{d^2}{dy^2}F\right)(k)\approx\left(F(k\underset{N}{\$}1)-2F(k)+F(k\underset{N}{\oplus}1)\right),$$

where $N=2\pi/\Delta x\Delta y$ and indices are taken modulo N. On the other hand,

$$\left(\mathrm{F}\left[P_{2\pi/\Delta x}D_{\Delta y}\frac{d^2}{dy^2}\right]\mathrm{F}^{-1}f\right)(n)\approx\left(\mathrm{F}\left[F(k\underset{N}{\$}1)-2F(k)+F(k\underset{N}{\oplus}1)\right]\mathrm{F}^{-1}\right)(n)=$$

$$=\left(f(n)e^{-\frac{2\pi i}{N}n}-2f(n)+f(n)e^{\frac{2\pi i}{N}n}\right)=2f(n)\left(\cos\left(\frac{2\pi}{N}n\right)-1\right).$$

These allow one to give the approximation for $\mathrm{H}=\frac{1}{2}\left(\frac{d^2}{dx^2}-x^2+1\right)$ as follows

$$\mathrm{H}ff(x)=\left[\frac{1}{2}\left(\frac{d^2}{dx^2}-x^2+1\right)\right]f(x)\approx$$

$$\approx\frac{1}{2}\left\{\left[f(n\underset{N}{\$}1)-2f(n)+f(n\underset{N}{\oplus}1)\right]+2f(n)\left(\cos\left(\frac{2\pi}{N}n\right)-1\right)+f(n)\right\}=$$

$$=-\left[\cos\frac{2\pi}{N}n-\frac{1}{2}\right]f(n)+\frac{1}{2}\left[f(n\underset{N}{\$}1)+f(n\underset{N}{\oplus}1)\right].$$

In the N-diagonal basis we have

$$\mathrm{F}^{d\alpha}f(x)\approx$$

$$\approx\begin{bmatrix}f(0)\\f(1)\\f(2)\\f(3)\\\vdots\\f(N-1)\end{bmatrix}+j\Delta\alpha\left[\begin{array}{c|c|c|c|c|c}-1/2 & 1/2 & . & . & . & 1/2\\\hline 1/2 & \cos(1\Omega)-3/2 & 1/2 & . & . & .\\\hline . & 1/2 & \cos(2\Omega)-3/2 & 1/2 & . & .\\\hline . & . & 1/2 & \cos(3\Omega)-3/2 & 1/2 & .\\\hline . & . & . & 1/2 & \ddots & 1/2\\\hline 1/2 & . & . & . & 1/2 & -1/2\end{array}\right]\begin{bmatrix}f(0)\\f(1)\\f(2)\\f(3)\\\vdots\\f(N-1)\end{bmatrix}, \tag{58}$$

where $\Omega=2\pi/N$.

Let us introduce the uniformly discretization (or sampling) of the angle parameter α on M discrete values $\{\alpha_0,\alpha_1,...,\alpha_i,\alpha_{i+1},...,\alpha_{M-1}\}$, where $\alpha_{i+1}=\alpha_i+\Delta\alpha$, $\alpha_i=i\Delta\alpha$. and $\Delta\alpha=2\pi/M$. Then

$$F^{\alpha_{i+1}}(y)=F^{\alpha_i+\Delta\alpha}(y)\approx F^{\alpha_i}(k)+j\Delta\alpha\times$$

$$\times\left\{\left[\cos\frac{2\pi}{N}k-\frac{3}{2}\right]F^{\alpha_i}(k)+\frac{1}{2}\left[F^{\alpha_i}(k\underset{N}{\$}1)+F^{\alpha_i}(k\underset{N}{\oplus}1)\right]\right\}. \tag{59}$$

It is easy to see that this algorithm requires $2MN$ multiplications and $3MN$ additions vs. $MN(2+\log_2 N)$ multiplications and $MN\log_2 N$ additions in the classical algorithm.

In (8) we used $O(h^2)$ approximation $\left(\frac{d^2}{dx^2}f\right)(k)\approx\left(f(k-1)-2f(k)+f(k+1)\right)$. Finer approximations $O(h^{2k})$ also can be used (Candan, 1998).

Conclusions

In this work, we introduce a new way of computing for Fractional Fourier transforms based on the infinitesimal Fourier transform. $2MN$ multiplications and $3MN$ additions are necessary vs. $MN(2+\log_2 N)$ multiplications and $MN\log_2 N$ additions in the classical algorithm. Presented algorithm can be utilized for fast computation in most applications of signal and image processing. We have presented a definition of the infinitesimal Fourier transform that exactly satisfies the properties of the Schrodinger Equation for quantum harmonic oscillator.

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