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FRÉCHET MIMO-FILTERS




Introduction

We develop a conceptual framework and design methodologies for multichannel image median filtering systems with assessment capability. The term multichannel (multicomponent, multispectral, hyperspectral) image is used for an image with more than one component. They are composed of a series of images in different optical bands at wavelengths $\lambda_1, \lambda_2, ..., \lambda_K$, called spectral channels: $\vec{\mathbf{f}}(x,y) = \left(f_{\lambda_1}(x,y), f_{\lambda_2}(x,y), ..., f_{\lambda_K}(x,y) \right)$, where K is the number of different optical channels, *i.e.*, $\mathbf{f}(x,y): \mathbf{R}^2 \to \mathbf{R}^K$, where $\mathbf{R}^K$ is multicolor space.

Let us introduce the observation model and notion used throughout the paper. We consider noise images of the $\mathbf{f}(\mathbf{x}) = \mathbf{s}(\mathbf{x}) + \boldsymbol{\eta}(\mathbf{x})$, where $\mathbf{s}(\mathbf{x})$ is the original K-channel image $\mathbf{s}(\mathbf{x}) = \left(s_1(\mathbf{x}), s_2(\mathbf{x}), ..., s_K(\mathbf{x}) \right)$ and $\boldsymbol{\eta}(\mathbf{x})$ denotes the K-channel noise $\boldsymbol{\eta}(\mathbf{x}) = \left(\eta_1(\mathbf{x}), \eta_2(\mathbf{x}), ..., \eta_K(\mathbf{x}) \right)$ introduced into the image $\mathbf{s}(\mathbf{x})$ to produce the corrupted image $\mathbf{f}(\mathbf{x}) = \left(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_K(\mathbf{x}) \right)$. Here, $\mathbf{x} = (i,j) \in \mathbf{Z}^2$ is a 2D coordinates. The aim of image enhancement is to reduce the noise as much as possible or to find a method, which, given $\mathbf{s}(\mathbf{x})$, derives an image $\hat{\mathbf{s}}(\mathbf{x})$ as close as possible to the original $\mathbf{s}(\mathbf{x})$ subjected to a suitable optimality criterion. In a 2D standard linear and median scalar filters with a square N-cellular window $\mathrm{M}_{(i,j)}(m,n)$ and located at (i,j), the mean and median replace the central pixel

$$\hat{s}(i,j) = \underset{(m,n)\in M_{(i,j)}}{\mathbf{Mean}} \left[f(m,n) \right], \tag{1}$$

$$\hat{s}(i,j) = \underset{(m,n)\in M_{(i,j)}}{\mathbf{Med}} \left[f(m,n) \right], \tag{2}$$

where $\hat{s}(i,j)$ is the filtered grey-level image, $\left\{ f(m,n) \right\}_{(m,n)\in M_{(i,j)}}$ is an image block of the fixed size N extracted from f by moving N-cellular window $\mathrm{M}_{(i,j)}$ at the position (i,j), $\mathbf{Mean}$ and $\mathbf{Med}$ are the mean (average) and median operators.

Median filtering has been widely used in image processing as an edge preserving filter. The basic idea is that the pixel value is replaced by the median of the pixels contained in a window around it. In this work, this idea is extended to vector-valued images, based on the fact that the median is also the value that minimizes the L_1 distance in $\mathbf{R}$ between all the gray-level pixels in the N-cellular window (Fig.1).

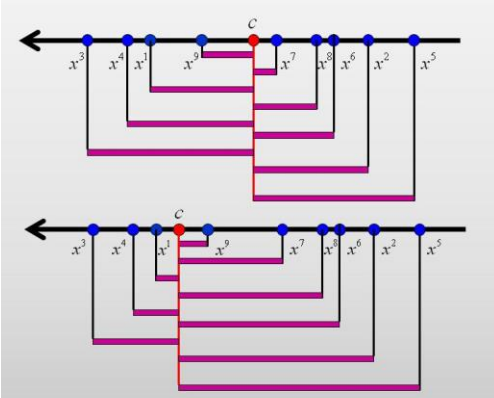

Fig.1. Distances from an arbitrary point c to each point $x^1, x^2, \dots, x^N \in \mathbf{R}$ from 9-cellular window.

In multichannel case, we need to define a distance ρ between pairs of objects on the domain $\mathbf{R}^K$. Let $\langle \mathbf{R}^K, \rho \rangle$ be a metric multicolour space, and $\rho(\mathbf{x},\mathbf{y})$ is a distance function for pair of objects $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^K$ (that is, $\rho(\mathbf{x},\mathbf{y}) : \mathbf{R}^K \times \mathbf{R}^K \to \mathbf{R}^+$). Let $w^1, w^2, \dots, w^N$ be N weights summing to 1 and let $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N \subset \mathbf{R}^K$ be N observations (for example, N pixels in the N-cellular window).

Definition 1 (Fréchet, 1948; Bajaj, 1986; 1988). *The optimal Fréchet point associated with the metric* $\rho(\mathbf{x},\mathbf{y})$ *is the point,* $\mathbf{c}_{opt} \in \mathbf{R}^K$ *that minimizes the Fréchet cost function* $\sum_{i=1}^{N} w_i \rho(\mathbf{c}, \mathbf{x}^i)$ *(the weighted sum distances from an arbitrary point* $\mathbf{c}$ *to each point* $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N$*). It is formally defined as*

$$\mathbf{c}_{opt} \equiv \mathbf{FrechPt}\left(\rho \mid \mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\right) = \mathbf{arg}\min_{\mathbf{c} \in \mathbf{R}^K} \left(\sum_{i=1}^{N} w_i \rho\left(\mathbf{c}, \mathbf{x}^i\right) \right). \quad (3)$$

Note that $\mathbf{arg}\min$ means the argument, for which the sum is minimized. In this case, it is the point $\mathbf{c}_{opt}$ from $\mathbf{R}^K$, for which the sum of all distances to the $\mathbf{x}^i$'s is minimum. So, the optimal Fréchet point of a discrete set of the observations (N pixels) in the metric space $\langle \mathbf{R}^K, \rho \rangle$ is the point minimizing the sum of distances to the N pixels (Fig. 2).

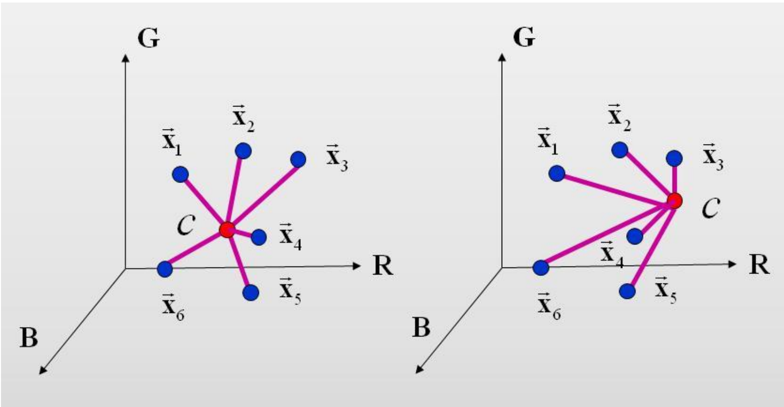

Fig. 2. Distances from an arbitrary point $\mathbf{c}$ to each point $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N \subset \mathbf{R}^K$ from 9-cellular window.

This generalizes the ordinary median, which has the property of minimizing the sum of distances for one-dimensional data. The properties of this point have been extensively studied since the time of Fermat, (this point is often called the *Fréchet point* (Fréchet, 1948) or *Fermat-Weber point* (Chandrasekaran, Tamir, 1990). In this paper, we extend the notion of Fréchet point to generalized Fréchet point which minimizes the aggregation Fréchet cost function (AFCF) in the form of an aggregation function ${}^{cf}\mathbf{Agg}_{i=1}^{N}\left[w_i \rho\left(\mathbf{c}, \mathbf{x}^i\right)\right]$, instead of the ordinary sum (3):

$$\mathbf{c}_{opt} \equiv \mathbf{FrechPt}\left({}^{cf}\mathbf{Agg}, \rho \,|\, \mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N\right) = \mathbf{arg}\min_{\mathbf{c}\in\mathbf{R}^K}\left({}^{cf}\mathbf{Agg}_{i=1}^N\left[w_i\rho\left(\mathbf{c}, \mathbf{x}^i\right)\right]\right). \quad (4)$$

Moreover, we propose use an aggregation distance ${}^{\rho}\mathbf{Agg}(\mathbf{c},\mathbf{x})$ instead of the classical distance ρ. It gives new cost function

$${}^{cf}\mathbf{Agg}\left[w_1\,{}^{\rho}\mathbf{Agg}\left(\mathbf{c},\mathbf{x}^1\right), w_2\,{}^{\rho}\mathbf{Agg}\left(\mathbf{c},\mathbf{x}^2\right), ..., w_N\,{}^{\rho}\mathbf{Agg}\left(\mathbf{c},\mathbf{x}^N\right)\right]$$

and new optimal Fréchet point associated with the aggregation distance ${}^{\rho}\mathbf{Agg}(\mathbf{c},\mathbf{x})$ and the aggregation Fréchet cost function ${}^{cf}\mathbf{Agg}$

$$\mathbf{c}_{opt} \equiv \mathbf{FrechPt}\left({}^{cf}\mathbf{Agg}, {}^{\rho}\mathbf{Agg} \,|\, \mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N\right) =$$
$$= \mathbf{arg}\min_{\mathbf{c}\in\mathbf{R}^K}\left({}^{cf}\mathbf{Agg}_{i=1}^N\left[w_i \cdot {}^{\rho}\mathbf{Agg}\left(\mathbf{c},\mathbf{x}^i\right)\right]\right). \quad (5)$$

We use generalized Fréchet point for constructing new nonlinear filters. When filters (1) are modified as follows:

$$\hat{\mathbf{s}}(i,j) = \mathbf{FrechPt}_{(m,n)\in M_{(i,j)}}\left[{}^{cf}\mathbf{Agg}, {}^{\rho}\mathbf{Agg} \,|\, \mathbf{f}(m,n)\right], \quad (6)$$

it becomes *Fréchet aggregation mean* filters. They are based on an arbitrary pair of aggregation operators ${}^{cf}\mathbf{Agg}$ and ${}^{\rho}\mathbf{Agg}(\mathbf{c},\mathbf{x})$, which could be changed independently of one another. For each pair of aggregation operators, we get the unique class of new nonlinear filters.

The suboptimal weighted Fréchet median

In computation point view, it is better to restrict the infinite search domain from $\mathbf{R}^K$ until the finite subset $\mathbf{D} = \left\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N\right\} \subset \mathbf{R}^K$. In this case, we obtain new definition of the *suboptimal Fréchet point* or the *optimal Fréchet median*.

Definition 2. *The suboptimal Fréchet point (or optimal Fréchet median) associated with the metric* $\rho(\mathbf{x},\mathbf{y})$ *is the point* $\hat{\mathbf{c}} \in \mathbf{D}$ *that minimizes the FCF over the the restricted search domain* $\mathbf{D} \subset \mathbf{R}^K$

$$\hat{\mathbf{c}}_{opt} \equiv \mathbf{FrechMed}\left(\rho \,|\, \mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N\right) = \mathbf{arg}\min_{\mathbf{c}\in\mathbf{D}}\left(\sum_{i=1}^N w_i\rho\left(\mathbf{c},\mathbf{x}^i\right)\right). \quad (7)$$

We use the generalized Fréchet point and median for constructing new nonlinear filters. When filters (1)-(2) are modified as follows:

$$\hat{s}(i,j) = \mathbf{FrechPt}_{(m,n)\in M_{(i,j)}}\left[\rho \,|\, f(m,n)\right], \quad (8)$$

$$\hat{s}(i,j) = \mathbf{FrecMed}_{(m,n)\in M_{(i,j)}}\left[\rho \,|\, f(m,n)\right] \quad (9)$$

it becomes *Fréchet mean* and *median filters*, associated with the metric $\rho(\mathbf{x},\mathbf{y})$.

Example 1. If observation data are real numbers, *i.e.*, $x^1, x^2, ..., x^N \in \mathbf{R}$, and the distance function is the city distance $\rho(x,y) = \rho_1(x,y) = |x-y|$, then the optimal Fréchet point (3) and median (7) for data $x^1, x^2, ..., x^N \in \mathbf{R}$ to be the classical *Fréchet point* and *classical median*, respectively. They are associated with the city metric $\rho_1(\mathbf{x},\mathbf{y})$, *i.e.*,

$$\mathbf{c}_{opt} = \mathbf{FrechPt}_{(m,n)\in M(i,j)}(\rho_1 \,|\, \mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N) = \mathbf{arg}\min_{\mathbf{c}\in\mathbf{R}}\left(\sum_{i=1}^N\left|c - x^i\right|\right), \quad (10)$$

$$\hat{c}_{opt} \equiv \mathbf{FrechMed}\left(\rho_1 \,|\, \mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N\right) =$$
$$= \mathbf{arg}\min_{\mathbf{c}\in\mathbf{D}}\left(\sum_{i=1}^N\left|c - x^i\right|\right) = \mathbf{Med}\left(x^1, x^2, ..., x^N\right). \quad (11)$$

In this case, filter (8) is *optimal maximum likelihood filter* for Laplace noise, and filter (9) is *ordinary median filter*.

Example 2. If observation data are vectors, *i.e.*, $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N \in \mathbf{R}^K$ and the distance function is the city distance $\rho(\mathbf{x},\mathbf{y}) = \rho_1(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|_1$, then the Fréchet point (3) and median (8) for vectors $\mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N \in \mathbf{R}^K$ to be the Fréchet point (vector) and vector median, respectively, associated with the same metric $\rho_1(\mathbf{x},\mathbf{y})$

$$c_{opt} \equiv \mathbf{FrechPt}\left(\rho_1 \mid \mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N\right) = \mathbf{arg}\min_{\mathbf{c}\in\mathbf{R}^K}\left(\sum_{i=1}^{N} \| \mathbf{c}-\mathbf{x}^i \|_1\right), \quad (12)$$

$$\hat{c}_{opt} \equiv \mathbf{FrechMed}\left(\rho_1 \mid \mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N\right) = \\ = \mathbf{arg}\min_{\mathbf{c}\in\mathbf{D}}\left(\sum_{i=1}^{N} \| \mathbf{c}-\mathbf{x}^i \|_1\right) = \mathbf{VecMed}\left(\rho_1 \mid \mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N\right). \quad (13)$$

In this case, filter (8) is *optimal maximum likelihood vector filter* for Laplace noise, and filter (9) is *vector median filter* associated with city metric (Astola et al., 1990; Tang et al., 1996).

Example 3. If observation data are vectors again: $\mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N \in \mathbf{R}^K$ but the distance function is the Euclidean $\rho(\mathbf{x},\mathbf{y}) = \rho_2(\mathbf{x},\mathbf{y}) = \| \mathbf{x}-\mathbf{y} \|$, then the Fréchet point (3), and median (7) for vectors $\mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N \in \mathbf{R}^K$ to be *Fréchet point (vector) and vector median,* respectively, associated with the Euclidean metric $\rho_2(\mathbf{x},\mathbf{y})$

$$c_{opt} \equiv \mathbf{FrechPt}\left(\rho_2 \mid \mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N\right) = \mathbf{arg}\min_{\mathbf{c}\in\mathbf{R}^K}\left(\sum_{i=1}^{N} \| \mathbf{c}-\mathbf{x}^i \|_2\right), \quad (14)$$

$$\hat{c}_{opt} \equiv \mathbf{FrechMed}\left(\rho_2 \mid \mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N\right) = \\ \mathbf{arg}\min_{\mathbf{c}\in\mathbf{D}}\left(\sum_{i=1}^{N} \| \mathbf{c}-\mathbf{x}^i \|_2\right) = \mathbf{VecMed}(\rho_2 \mid \mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N). \quad (15)$$

In this case, filter (8) is *optimal maximum likelihood vector filter* for Gaussian noise, and filter (9) is vector median filter associated with Euclidean metric.

Generalized vector aggregation

In Definitions 1 and 2, the Fréchet point and median are points $\mathbf{c}_{opt} \in \mathbf{R}^K$, $\hat{\mathbf{c}}_{opt} \in \mathbf{D} = \left\{\mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N\right\}$ that minimize the Fréchet cost function (FCF) $\sum_{i=1}^{N} w_i \rho\left(\mathbf{c},\mathbf{x}^i\right)$. But this sum up to constant factor is the simplest aggregation function.

The aggregation problem consist in aggregating *n*-tuples of objects all belonging to a given set S, into a single object of the same set S, i.e., $\mathbf{Agg}: S^n \to S$. In the case of mathematical aggregation operator the set S is an interval of the real $S=[0,1]\subset\mathbf{R}$ or integer numbers $S=[0,255]\subset\mathbf{Z}$. In this setting, an AO is simply a function, which assigns a number y to any N-tuple $(x_1,x_2,...,x_N)$ of numbers: $y = \mathbf{Aggreg}(x_1,x_2,...,x_N)$ that satisfies:

- $\mathbf{Agg}(x) = x,$
- $\mathbf{Agg}(\underbrace{a,a,...,a}_{N}) = a.$ In particular, $\mathbf{Agg}(0,0,...,0) = 0$ and $\mathbf{Agg}(1,1,...,1) = 1,$ or

  $\mathbf{Agg}(255,255,...,255) = 255.$
- $\mathbf{min}(x_1,x_2,...,x_N) \le \mathbf{Agg}(x_1,x_2,...,x_N) \le \mathbf{max}(x_1,x_2,...,x_N).$

Here $\mathbf{min}(x_1,x_2,...,x_N)$ and $\mathbf{max}(x_1,x_2,...,x_N)$ are respectively the *minimum* and the *maximum* values among the elements of $(x_1,x_2,...,x_N)$.

All other properties may come in addition to this fundamental group. For example, if for every permutation $\forall\sigma\in\mathbf{S}_N$ of $\{1,2,...,N\}$ the AO satisfies:

$$\mathbf{Agg}\left(x_{\sigma(1)},x_{\sigma(2)},...,x_{\sigma(N)}\right) = \mathbf{Agg}(x_1,x_2,...,x_N),$$

then it is invariant (symmetric) with respect to the permutations of the elements of $(x_1,x_2,...,x_N)$. In other words, as far as means are concerned, the *order* of the elements of $(x_1,x_2,...,x_N)$ is - and must be - completely irrelevant.

According to Kolmogorov (1930) a sequence of functions $\mathbf{Agg}_N(x_1, x_2, ..., x_N)$ (for different N) defines a regular type of average if the following conditions are satisfied:

1) $\mathbf{Agg}_N(x_1, x_2, ..., x_N)$ is continuous and monotone in each variable; to be definite, we assume that $\mathbf{Agg}_N$ is increasing in each variable.

2) $\mathbf{Agg}_N(x_1, x_2, ..., x_N)$ is a symmetric function.

3) The average of identical numbers is equal to their common value: $\mathbf{Agg}_N(x, x, ..., x) = x$.

4) A group of values can be replaced by their own average, without changing the overall average:

$$\mathbf{Agg}_{N+M}(x_1, x_2, ..., x_N, y_1, y_2, ..., y_M) = \mathbf{Agg}_{N+M}(m, m, ..., m, y_1, y_2, ..., y_M),$$

where $m = \mathbf{Agg}_N(x_1, x_2, ..., x_N)$.

Proposition 1. (Kolmogorov, 1930). If conditions (1)–(4) are satisfied, the average $\mathbf{Agg}_N(x_1, x_2, ..., x_N)$ is of the form:

$$\mathbf{Kolm}(K \mid x_1, x_2, ..., x_N) = K^{-1}\left[\sum_{i=1}^{N} \frac{1}{N} K(x_i)\right],$$

where K is a strictly monotone continuous function in the extended real line.

We list below a few particular cases of means:

1) Arithmetic mean ($K(x) = x$): $\mathbf{Mean}(x_1, x_2, ..., x_N) = \frac{1}{N}\sum_{i=1}^{N} x_i$.

2) Geometric mean ($K(x) = \log(x)$): $\mathbf{Geo}(x_1, x_2, ..., x_N) = \exp\left(\frac{1}{N}\sum_{i=1}^{N} \ln x_i\right)$.

3) Harmonic mean ($K(x) = x^{-1}$): $\mathbf{Harm}(x_1, x_2, ..., x_N) = \dfrac{1}{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{x_i}} = \dfrac{N}{\sum_{i=1}^{N}\frac{1}{x_i}}$.

4) A very notable particular case corresponds to the function $K(x) = x^p$. We obtain then a quasi arithmetic (power or Hölder) mean of the form: $\mathbf{Power}_p(x_1, x_2, ..., x_N) = \left(\frac{1}{N}\sum_{i=1}^{N} x_i^p\right)^{\frac{1}{p}}$.

This family is particularly interesting, because it generalizes a group of common means, only by changing the value of p.

A very notable particular cases correspond to the logic functions (min; max; median): $y = \mathbf{Min}(x_1, x_2, ..., x_N)$, $y = \mathbf{Max}(x_1, x_2, ..., x_N)$, $y = \mathbf{Med}(x_1, x_2, ..., x_N)$.

When filters (1) and (2) are modified as follows:

$$\hat{s}(i, j) = \mathop{\mathbf{Agg}}_{(m,n)\in M_{(i,j)}}\left[f(m,n)\right], \tag{16}$$

we get the unique class of nonlinear aggregation filters proposed in the works (Labunets, 2014; Labunets et al., 2014 a,b,c).

In this work, we are going to use the cost function in the form of an aggregation function

$${}^{cf}\mathbf{Agg}_{i=1}^{N}\left[w_i\rho\left(\mathbf{c}, \mathbf{x}^i\right)\right] = {}^{cf}\mathbf{Agg}\left[w_1\rho\left(\mathbf{c}, \mathbf{x}^1\right), w_2\rho\left(\mathbf{c}, \mathbf{x}^2\right), ..., w_N\rho\left(\mathbf{c}, \mathbf{x}^N\right)\right]$$

instead of

$$\sum_{i=1}^{N} w_i\rho\left(\mathbf{c}, \mathbf{x}^i\right) = \left[w_1\rho\left(\mathbf{c}, \mathbf{x}^1\right), w_2\rho\left(\mathbf{c}, \mathbf{x}^2\right), ..., w_N\rho\left(\mathbf{c}, \mathbf{x}^N\right)\right].$$

We obtain the next generalization of the Fréchet point and median.

Definition 3. *The Fréchet aggregation point and median are the points* $\mathbf{c}_{opt} \in \mathbf{R}^K$ *and* $\hat{\mathbf{c}}_{opt} \in D = \left\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N\right\}$ *that minimize the aggregation cost function (ACF)* ${}^{cf}\mathbf{Agg}_{i=1}^{N}\left[w_i\rho\left(\mathbf{c}, \mathbf{x}^i\right)\right]$. *They are formally defined as*

$$\mathbf{c}_{opt} = \mathbf{FrechPt}\left({}^{cf}\mathbf{Agg}, \rho \mid \mathbf{x}^1, ..., \mathbf{x}^N\right) = \mathop{\mathbf{arg\,min}}_{\mathbf{c}\in\mathbf{R}^K}\left({}^{cf}\mathbf{Agg}_{i=1}^{N}\left[w_i\rho\left(\mathbf{c}, \mathbf{x}^i\right)\right]\right) \tag{17}$$

and

$$\hat{\mathbf{c}}_{opt} \equiv \mathbf{FrechMed}\left({}^{cf}\mathbf{Agg}, \rho \,|\, \mathbf{x}^1,...,\mathbf{x}^N\right) = \underset{\mathbf{c}\in\mathbf{D}}{\mathbf{arg}\min}\left({}^{cf}\mathbf{Agg}_{i=1}^N\left[w_i\rho\left(\mathbf{c},\mathbf{x}^i\right)\right]\right). \quad (18)$$

Note that **arg**min means the argument, for which the ${}^{cf}\mathbf{Agg}_{i=1}^N\left[w_i\rho\left(\mathbf{c},\mathbf{x}^i\right)\right]$ is minimized. In this case, it is the point $\mathbf{c}_{opt}\in\mathbf{R}^K$ in (17) or point $\hat{\mathbf{c}}_{opt}\in\mathbf{D}=\left\{\mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N\right\}$ in (18) for which the aggregation of all distances to the $\mathbf{x}^i$'s is minimum.

When filters (1)--(2) are modified as follows:

$$\hat{\mathbf{s}}(i,j) = \underset{(m,n)\in M_{(i,j)}}{\mathbf{FrechPt}}\left[{}^{cf}\mathbf{Agg}, \rho \,|\, \mathbf{f}(m,n)\right], \quad (19)$$

$$\hat{\mathbf{s}}(i,j) = \underset{(m,n)\in M_{(i,j)}}{\mathbf{FrechMed}}\left[{}^{cf}\mathbf{Agg}, \rho \,|\, \mathbf{f}(m,n)\right] \quad (20)$$

it becomes *Fréchet aggregation mean* and *median filters*. They are based on an aggregation operator ${}^{cf}\mathbf{Agg}$ and a metric ρ, which could be changed independently of one another. For each pair of aggregation operator and metric, we get the unique class of new nonlinear filters.

Example 4. If observation data are real numbers, *i.e.*, $x^1,x^2,...,x^N\in\mathbf{R}$, the distance function is the city distance $\rho(x,y)=\rho_1(x,y)=|x-y|$ and ACF is quadratic ${}^{cf}\mathbf{Agg}_{i=1}^N\left[w_i\rho\left(\mathbf{c},\mathbf{x}^i\right)\right]=\sum_{i=1}^N w_i\rho_1^2\left(\mathbf{c},\mathbf{x}^i\right)$, then the optimal Fréchet point (17) and median (18) for grey-level data (numbers) $x^1,x^2,...,x^N\in\mathbf{R}$ to be the ordinary arithmetic mean, and quadratic median, respectively, *i.e.*,

$$\begin{aligned}\mathbf{c}_{opt} &\equiv \mathbf{FrechPt}\left({}^{cf}\mathbf{Agg}, \rho_2 \,|\, \mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N\right) = \\ &= \underset{\mathbf{c}\in\mathbf{R}^K}{\mathbf{arg}\min}\left(\sum_{i=1}^N\left|c-x^i\right|^2\right) = \mathbf{Mean}\left(x^1,x^2,...,x^N\right) = \frac{1}{N}\sum_{i=1}^N x^i.\end{aligned} \quad (21)$$

$$\hat{\mathbf{c}}_{opt} \equiv \mathbf{FrechMed}\left({}^{cf}\mathbf{Agg}, \rho \,|\, \mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N\right) = \underset{\mathbf{c}\in\mathbf{D}}{\mathbf{arg}\min}\left(\sum_{i=1}^N\left|c-x^i\right|^2\right). \quad (22)$$

In this case, filter (19) is *optimal maximum likelihood vector filter* for Gaussian noise, and filter (20) is *vector median filter* associated with Euclidean metric, because $\rho_2(x,y)=\rho_1^2(x,y)=|x-y|^2$..

Example 5. If observation data are vectors, *i.e.*, $\mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N\in\mathbf{R}^K$, the distance function is the city distance $\rho_1(\mathbf{x},\mathbf{y})=\|\mathbf{x}-\mathbf{y}\|_1$, and ACF is the Kolmogorov mean

$${}^{cf}\mathbf{Agg}_{i=1}^N\left[w_i\rho\left(\mathbf{c},\mathbf{x}^i\right)\right] = K^{-1}\left[\sum_{i=1}^N w_iK\left(\|\mathbf{c}-\mathbf{x}\|_1\right)\right]$$

then Fréchet aggregation point and median for vectors $\mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N\in\mathbf{R}^K$ to be the following Kolmogorov-Fréchet aggregations

$$\mathbf{c}_{opt} \equiv \mathbf{FrechPt}\left(K, \rho_1 \,|\, \mathbf{x}^1,...,\mathbf{x}^N\right) = \underset{\mathbf{c}\in\mathbf{R}^K}{\mathbf{arg}\min}\left(K^{-1}\left[\sum_{i=1}^N w_iK\left(\|\mathbf{c}-\mathbf{x}\|_1\right)\right]\right), \quad (23)$$

$$\hat{\mathbf{c}}_{opt} \equiv \mathbf{FrechMed}\left(K, \rho_1 \,|\, \mathbf{x}^1,...,\mathbf{x}^N\right) = \underset{\mathbf{c}\in\mathbf{D}}{\mathbf{arg}\min}\left(K^{-1}\left[\sum_{i=1}^N w_iK\left(\|\mathbf{c}-\mathbf{x}\|_1\right)\right]\right), \quad (24)$$

respectively. In this case, filter (19) is the *Kolmogorov--Fréchet vector mean filter*, and filter (20) is the *Kolmogorov-Fréchet vector median filter* associated with sity metric.

In Definitions 1 and 2 we used a distance function $\rho(\mathbf{x},\mathbf{y})$. But all known metrics have the aggregation form. By this reason, we can use an aggregation function ${}^{\rho}\mathbf{Agg}\left(|c_1-x_1|,...,|c_K-x_K|\right)$ instead of $\rho\left(|c_1-x_1^i|,...,|c_K-x_K^i|\right)$.

Definition 4. *The Fréchet aggregation point and median are the points* $\mathbf{c}_{opt}\in\mathbf{R}^K$ *and* $\hat{\mathbf{c}}_{opt}\in\mathbf{D}=\left\{\mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^N\right\}$ *that minimizes the aggregation cost function (ACF)* ${}^{cf}\mathbf{Agg}\left(w_1\,{}^{\rho}\mathbf{Agg}\left(\mathbf{c},\mathbf{x}^1\right),...,w_N\,{}^{\rho}\mathbf{Agg}\left(\mathbf{c},\mathbf{x}^N\right)\right)$ *(=the weighted aggregation mean of all aggregation dis-*

tances $w_1\,{}^{\rho}\mathbf{Agg}(\mathbf{c},\mathbf{x}^1), w_2\,{}^{\rho}\mathbf{Agg}(\mathbf{c},\mathbf{x}^2),\ldots\ \ldots, w_N\,{}^{\rho}\mathbf{Agg}(\mathbf{c},\mathbf{x}^N)$ *from an arbitrary point* $\mathbf{c}\subset\mathbf{R}^K$ *to each point* $\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^N\in\mathbf{R}^K$ *). They are formally defined as*

$$\begin{aligned}\mathbf{c}_{opt} &\equiv \mathbf{FrechPt}\left({}^{cf}\mathbf{Agg}, {}^{\rho}\mathbf{Agg} \mid \mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^N\right) = \\ &= \mathbf{arg}\min_{\mathbf{c}\in\mathbf{R}^K}\left({}^{cf}\mathbf{Agg}_{i=1}^N\left[w_i\cdot{}^{\rho}\mathbf{Agg}(\mathbf{c},\mathbf{x}^i)\right]\right)\end{aligned} \tag{25}$$

and

$$\begin{aligned}\hat{\mathbf{c}}_{opt} &\equiv \mathbf{FrechMed}\left({}^{cf}\mathbf{Agg}, {}^{\rho}\mathbf{Agg} \mid \mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^N\right) = \\ &= \mathbf{arg}\min_{\mathbf{c}\in\mathbf{D}}\left({}^{cf}\mathbf{Agg}_{i=1}^N\left[w_i\cdot{}^{\rho}\mathbf{Agg}(\mathbf{c},\mathbf{x}^i)\right]\right).\end{aligned} \tag{26}$$

We use generalized Fréchet point for constructing new nonlinear filters. When filters (1) are modified as follows:

$$\hat{\mathbf{s}}(i,j) = \mathbf{FrechPt}_{(m,n)\in M_{(i,j)}}\left[{}^{cf}\mathbf{Agg}, {}^{\rho}\mathbf{Agg} \mid \mathbf{f}(m,n)\right], \tag{27}$$

$$\hat{\mathbf{s}}(i,j) = \mathbf{FrechMed}_{(m,n)\in M_{(i,j)}}\left[{}^{cf}\mathbf{Agg}, {}^{\rho}\mathbf{Agg} \mid \mathbf{f}(m,n)\right] \tag{28}$$

it becomes Fréchet aggregation mean and median filters. They are based on an arbitrary pair of aggregation operators ${}^{cf}\mathbf{Agg}$ and ${}^{\rho}\mathbf{Agg}(\mathbf{c},\mathbf{x})$, which could be changed independently of one another. For each pair of aggregation operators, we get the unique class of new nonlinear filters.

Example 6. For vector observation data $\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^N\in\mathbf{R}^K$, for L-Kolmogorov aggregation distance function $L^{-1}\left(\sum_{k=1}^K L(|c_k - x_k^i|)\right)$, and for ACF in the form of the K-Kolmogorov mean $K^{-1}\left(\sum_{i=1}^N w_i K\left[{}^{\rho}\mathbf{Agg}(\mathbf{c},\mathbf{x}^i)\right]\right)$, we have

$$\begin{aligned}\mathbf{c}_{opt} &\equiv \mathbf{FrechPt}\left(K,L \mid \mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^N\right) = \mathbf{arg}\min_{\mathbf{c}\in\mathbf{R}^K}\left({}^{cf}\mathbf{Agg}_{i=1}^N\left[w_i\,{}^{\rho}\mathbf{Agg}(\mathbf{c},\mathbf{x}^i)\right]\right) = \\ &= \mathbf{arg}\min_{\mathbf{c}\in\mathbf{R}^K}\left\{K^{-1}\left(\sum_{i=1}^N w_i K\left[L^{-1}\left(\sum_{k=1}^K L(|c_k - x_k^i|)\right)\right]\right)\right\},\end{aligned} \tag{29}$$

$$\begin{aligned}\hat{\mathbf{c}}_{opt} &\equiv \mathbf{FrechMed}\left(K,L \mid \mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^N\right) = \mathbf{arg}\min_{\mathbf{c}\in\mathbf{R}^K}\left({}^{cf}\mathbf{Agg}_{i=1}^N\left[w_i\,{}^{\rho}\mathbf{Agg}(\mathbf{c},\mathbf{x}^i)\right]\right) = \\ &= \mathbf{arg}\min_{\mathbf{c}\in\mathbf{R}^K}\left\{K^{-1}\left(\sum_{i=1}^N w_i K\left[L^{-1}\left(\sum_{k=1}^K L(|c_k - x_k^i|)\right)\right]\right)\right\},\end{aligned} \tag{30}$$

where L and K are two Kolmogorov functions. In this case, filter (27) is the Kolmogorov--Fréchet vector mean filter, and filter (28) is the Kolmogorov--Fréchet vector median filter associated with sity metric and with a pair of Kolmogorov $K-, L-$ functions .

Experiments

Generalized vector aggregation filtering with ${}^{cf}\mathbf{Agg} = \mathbf{Mean}, \mathbf{Med}, \mathbf{Geo}$ has been applied to noised 256x256 image "*Dog*" (Figures 3b,4b,5b). We use window with size 3×3. The denoised images are shown in Figures 5-7. All filters have very good denoised properties.

Conclusions

A new class of nonlinear generalized MIMO-filters (vector median filters or Fréchet filters) for multichannel image processing is introduced in this paper. These filters are based on an arbitrary pair of aggregation operators, which could be changed independently of one another. For each pair of parameters, we get the unique class of new nonlinear filters. The main goal of the work is to show that generalized Fréchet aggregation means can be used to solve problems of image filtering in a natural and effective manner.

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a) Original image

b) Noised images, PSNR = 21.83

c) ${}^{cf}\mathbf{Agg} = \mathbf{Mean}$, ${}^{\rho}\mathbf{Agg} = \rho_2$, PSNR = 32.52

d) ${}^{cf}\mathbf{Agg} = \mathbf{Med}$, ${}^{\rho}\mathbf{Agg} = \rho_2$, PSNR = 31.79,

e) ${}^{cf}\mathbf{Agg} = \mathbf{Min}$, ${}^{\rho}\mathbf{Agg} = \rho_2$, PSNR = 28.29

f) ${}^{cf}\mathbf{Agg} = \mathbf{Geo}$, ${}^{\rho}\mathbf{Agg} = \rho_2$, PSNR = 30.52

Fig. 3. Original (*a*) and noised (*b*) images; noise: Salt-Pepper; denoised images (*c*)-(*f*).

a) Original image

b) Noised images, PSNR = 17,19

c) ${}^{cf}\mathbf{Agg} = \mathbf{Mean}$, ${}^{\rho}\mathbf{Agg} = \rho_2$, PSNR = 21.83

d) ${}^{cf}\mathbf{Agg} = \mathbf{Med}$, ${}^{\rho}\mathbf{Agg} = \rho_2$, PSNR = 20.84

e) $^{cf}\mathbf{Agg} = \mathbf{Min}$, $^{\rho}\mathbf{Agg} = \rho_2$, PSNR = 19.05

f) $^{\rho}\mathbf{Agg} = \mathbf{Geo}$, $^{\rho}\mathbf{Agg} = \rho_2$, PSNR = 20.51

Fig. 4. Original (*a*) and noised (*b*) images; noise: Gaussian PDF; de-noised images (*c*)-(*f*).

a) Original image

b) Noised images, PSNR = 28.24

c) $^{cf}\mathbf{Agg} = \mathbf{Mean}$, $^{\rho}\mathbf{Agg} = \rho_2$, PSNR = 30.68

d) $^{cf}\mathbf{Agg} = \mathbf{Med}$, $^{\rho}\mathbf{Agg} = \rho_2$, PSNR = 29.61

e) $^{cf}\mathbf{Agg} = \mathbf{Min}$, $^{\rho}\mathbf{Agg} = \rho_2$, PSNR = 27.77

f) $^{cf}\mathbf{Agg} = \mathbf{Geo}$, $^{\rho}\mathbf{Agg} = \rho_2$, PSNR = 30.14

Fig. 5. Original (*a*) and noised (*b*) images; noise: Laplasian PDF; denoised images (*c*)-(*f*).

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