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***V.G. Labunets*[1]*, V.P. Chasovskikh, E. Ostheimer*[2]**

[1] Ural State Forest Engineering University, Yekaterinburg
[2]Capricat LLC 1340 S. Ocean Blvd., Suite 209 Pompano Beach, 33062 Florida, USA

GENERALIZED CLASSICAL AND QUANTUM SIGNAL THEORIES ON HYPERGROUPS.
PART 1. CLASSICAL SIGNAL THEORY



Introduction

The integral transforms and the signal representation associated with them are important concepts in applied mathematics and in the signal theory. The Fourier transforms on Abelian groups are certainly the best known of the integral transforms. The use of Abelian groups in the signal theory is not new. The classical harmonic analysis of 1-D continuous signals and systems is the Fourier analysis on the additive group $\Omega = \mathbf{AR}$ of the real field $\mathbf{R}$. Finite discrete Fourier analysis in digital signal processing is associated with finite cyclic groups $\Omega = \mathbf{Z}/\mathrm{N}$. The rich and beautiful theory of harmonic analysis on $\Omega = \mathbf{AR}$ and $\Omega = \mathbf{Z}/\mathrm{N}$ has became a powerful tool, widely used in other branches of mathematics, in physics and in innumerable applications etc.

The fundamental properties of the classical Fourier transforms are actually based on the properties these two groups. From mathematical point of view, a significant part of classical digital signal processing (DSP) can be viewed as topics in Abelian group harmonic analysis. The classical Fourier transformations are closely connected to such powerful concepts of the signal theory as linear and nonlinear convolutions (Voltera convolutions), classical and high-order correlations, invariance with respect to shift, ambiguity and Wigner distributions, etc. All theorems and properties of this harmonic analysis can be transferred on harmonic analysis Fourier on arbitrary Abelian groups.

Fourier transforms on Abelian and non-Abelian groups is just one of many ways of signal representations. In the past 10 years, other analytical methods have been proposed and applied, for example, Hermite, Lagerra, Legendre, Gabor, fractional Fourier analysis, etc. An important aspect of many of these representations is the possibility to extract relevant information from a signal: information that is actually present but hidden in its complex representation.

The next modest idea in development of signal representations is using so-called multiparametric transforms (MPTs) having fast algorithms instead of concrete fixed transforms (for example, Fourier, Walsh, Haar, wavelet). The family of MPTs includes such subfamilies as multiparametric fractional Fourier transforms, multiparametric fractional Walsh transforms, multiparametric Haar-Wavelet transforms, multiparametric nonlinear

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transforms and so on. MPT depend on one or several free parameters. When parameters are changed in some range the type and form of transform are changed too. This fact allows to calculating spectra of signal/image for infinite number of transforms. The main purpose of using MPT is to select the best transform among of all MPTs, given a signal/image and an additive cost function.

There exist many generalization of harmonic analysis on Abelian and non-Abelian groups, based on variation ways of viewing Ω. We generalized the basic notations and results from harmonic analysis on groups to nonharmonic analysis on hypergroups. The aim of this work is to obtain analogues of classical and quantum harmonic notions and results for hypergroups, associated with arbitrary nonharmonic transforms (including manyparametric transforms). We develop a conceptual framework and design methodologies for nonharmonic analysis of signals and images on Abelian hypergroups.

Generalized shift operators

Most specialists are probably familiar with the special role played by translations in digital signal/image processing (DSIP). The translations are viewed as the primary source of classical DSIP operations including convolution, correlation, Wigner distribution. The translation-invariance of some classical signal/image processing transforms and filtering operations is largely responsible for their widespread use. The main problem to be faced in extending classical DSIP theory (associated with classical Fourier transform) on generalized DSIP theory (associated with arbitrary discrete transform) is to decide on what is meant by translation.

The ordinary group shift operators $\left(T_t^{\tau} x\right)(t) = x(t+\tau)$ play the leading role in all the properties and tools of the Fourier transform mentioned above. In order to develop for each orthogonal transform a similar wide set of tools and properties as the Fourier transform has, we associate a family of commutative generalized shift operators (GSO) with each orthogonal (unitary) transform. Such families form *hypergroups*. In 1934 F. Marty (1934, 1935) and H.S. Wall (1934, 1937) independently introduced the notion of hypergroup. We develop the theory of hyperharmonic analysis of signals and images. Such families of operators allow to unify and generalize the majority of known methods and tools of signal processing based on classical Fourier transform for generalized classical and quantum signal theories. Only in particular cases these families are Abelian groups and hyperharmonic analysis is the classical Fourier harmonic analysis on groups.

Let $f(x):\Omega \to \mathbf{A}$ be a $\mathbf{A}$-valued signal, where $\mathbf{A}$ be an algebra. It can be real, complex, Galois fields or triplet, multiplet or Clifford algebras. Usually, $\Omega = \mathbf{R}^n \times \mathbf{T}$, or $\Omega = \mathbf{Z}^n \times \mathbf{T}$, where $\mathbf{R}^n$, $\mathbf{Z}^n$ and $\mathbf{Z}_N^n$ are n-D vector spaces over **R**, **Z** and $\mathbf{Z}_N$, respectively, **T** is a compact (temporal) subset of **R**, **Z**, or $\mathbf{Z}_N$. Here, **R**, **Z** and $\mathbf{Z}_N$ are the real field, the ring of integers, and the ring of integers modulo N, respectively.

Let Ω^* be the space dual to Ω. The first one will be called the *spectral domain*, the second one be called *signal domain* keeping the original notion of $x \in \Omega$ as «time» and $\omega \in \Omega^*$ as «frequency». Let

$$\mathbf{ClSig} = L(\Omega, \mathbf{A}) := \left\{ f(x) \middle| f(x) : \Omega \to \mathbf{A} \right\},$$

$$\mathbf{ClSp} = L(\Omega^*, \mathbf{A}) := \left\{ F(\omega) \middle| F(\omega) : \Omega^* \to \mathbf{A} \right\}$$

be two vector spaces of $\mathbf{A}$-valued functions. Let $\{\varphi_\omega(x)\}_{\omega\in\Omega^*}$ be an orthonormal system of functions of $\mathbf{ClSig}$. Then for any function $f(x) \in \mathbf{ClSig}$ there exists such a function $F(\omega) \in \mathbf{ClSp}$, for which the following equations hold:

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$$F(\omega)=\mathcal{CF}\ \{f\}(\omega)=\int_{x\in\Omega} f(x)\overline{\varphi}_{\omega}(x)d\mu(x), \quad (1)$$

$$f(x)=\mathcal{CF}^{-1}\{F\}(x)=\int_{x\in\Omega^*} F(\omega)\varphi_{\omega}(x)d\mu(\omega), \quad (2)$$

where $\mu(x),\mu(\omega)$ are certain suitable measures on the signal Ω and spectral Ω^* domains, respectively. The function $F(\omega)$ is called the classical Fourier spectrum ($\mathcal{CF}$ -spectrum) of the classical $\mathbf{A}$ -valued signal $f(x)$ and expressions (1)-(2) are called the pair of *generalized classical Fourier transforms* (or $\mathcal{CF}$ -transforms). In the following we will use the notation $f(x)\underset{\mathcal{CF}}{\leftrightarrow}F(\omega)$ in order to indicate $\mathcal{CF}$ -transforms pair.

Remark. Every classical Fourier transform can has two realizations: a classical realization on classical computer and a quantum realization on quantum computer. Now, quantum realizations of a classical Fourier transform are called wrongly quantum Fourier transforms. We shall show that every classical Fourier transform generates its the quantum counterpart as a natural quantum Fourier transform. It maps classical word on quantum word. Every quantum Fourier transform can also has two realizations: a classical and quantum realizations, too.

If $\left\{\phi_{\omega}\left(x|\theta\right)\right\}_{\omega\in\Omega^*}$ is an orthonormal system of functions, depending on one parameter θ or several parameters $\theta=\left(\theta_1,\theta_2,\ldots,\theta_r\right)$ then associated classical $\mathcal{CF}$ -transform is called the multiparameter transform $\mathcal{CF}=\mathcal{CF}[\theta]$. For this reason, the transform (1) gives multiparameter spectrum

$$F\left(\omega|\theta\right)=\mathcal{CF}\ [\theta]\{f\}(\omega):=\int_{x\in\Omega} f(x)\varphi\left(\omega|\theta\right)d\mu(x). \quad (3)$$

Obviously,

$$f\left(x\right)=\mathcal{CF}^{-1}[\theta]\{F\}(x)=\int_{x\in\Omega^*} F\left(\omega|\theta\right)\varphi\left(\omega|\theta\right)d\mu(\omega)\,. \quad (4)$$

When parameters are changed in some range, the type and form of transform are changed too and we calculate spectra of signal/image for continue number of transforms belonging to MPT. All set of spectra is called the *spectrogram*. If in classical signal/image processing systems we take a look at single «photo» of spectrum (for example, the spectrum of an image in Fourier basis), then in multiparameter case we can look through the thriller «SPECTRA OF SIGNAL». For all of transforms of the same family of MPTs we can select any concrete transform (for fixed values of parameters). The output of a computation can be memorized to be used in later numerical computations. The main purpose of using MPT is to select the best transform among of all MPTs, given a signal/image and an additive cost function.

For example, let $\Omega=\left[-1,+1\right]$ and $\varphi_{\omega}\left(x|\theta\right)=Jac_k^{(\alpha,\beta)}(x)$ be $\left(\alpha,\beta\right)$-Jacobi polynomials, where $\theta=\left(\alpha,\beta\right)$. In this case, generalized classical Fourier transform is $\left(\alpha,\beta\right)$-parametric Fourier-Jacobi transform

$$F\left(k\left|\left(\alpha,\beta\right)\right.\right)=\mathcal{CF}\ \left[\left(\alpha,\beta\right)\right]\{f\}(k)=\int_{-1}^{+1} f(x)Jac_k^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1-x)^{\beta}dx\,,$$

where $d\mu(x)=(1-x)^{\alpha}(1-x)^{\beta}dx$.

If $\alpha=\beta=0$ then $\left\{Jac_k^{(0,0)}(x)\right\}_{k=0}^{\infty}=\left\{Leg_k(x)\right\}_{k=0}^{\infty}$ is the Legendre basis and $\mathcal{CF}\ [(0,0)]$ is the Legendre transform. If $\alpha=\beta=-0.5$, then $\left\{Jac_k^{(-0.5,-0.5)}\left(t\right)\right\}_{k=0}^{\infty}=\left\{Ch_k\left(t\right)\right\}_{k=0}^{\infty}$ is the Chebyshev basis and $\mathcal{CF}\ [(-0.5,-0.5)]$ is the Chebyshev transform.

The next example of MPT. Let

$$\mathcal{CF}\ \left[\left(\theta_1,\theta_2,\ldots,\theta_n\right)\right]:=cs\left(\theta_1\right)\otimes cs\left(\theta_2\right)\otimes\ldots\otimes cs\left(\theta_n\right)$$

Электронный архив УГЛТУ

be n-parametric orthogonal transform, where $cs(\theta)=\begin{bmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{bmatrix}$. When $\theta_1=\ldots=\theta_n=\pi/4$ it is the classical Walsh transform. For other values of $\theta_1,\theta_2\ldots,\theta_n$ we obtain other orthogonal transforms.

The classical shift operators in the «time» and «frequency» domains are defined as

$$\left(\overset{\tau}{T_x} f\right)(x):=f(x+\tau) \text{ and } \left(\overset{\nu}{D_\omega} F\right)(\omega):=F(\omega+\nu).$$

For $f(x)=e^{j\omega x}$ and $F(\omega)=e^{-j\omega x}$ we have

$$\widehat{T}_x^\tau e^{j\omega x}=e^{j\omega(x+\tau)}=e^{j\omega x}e^{j\omega\tau} \text{ and } \widehat{D}_\omega^\nu e^{-j\omega x}=e^{-j(\omega+\nu)}=e^{-j\nu x}e^{-j\omega x},$$

i.e., harmonic signals $e^{j\omega x}$, $e^{-j\omega x}$ are eigenfunctions of «time»-shift and «frequency»-shift operators $\overset{\tau}{T_x}$ and $\overset{\nu}{D_\omega}$.

We introduce generalized θ-parametrized «time»-shift and «frequency»-shift operators (GSOs) by

$$\overset{\tau}{T_x}[\theta]f(x):=f\left(x \underset{\theta}{(} \tau\right), \qquad \overset{\bar{\tau}}{T_x}[\theta]f(x):=f\left(x \underset{\theta}{'} \tau\right),$$

$$\overset{\nu}{D_\omega}[\theta]F(\omega|\theta):=F\left(\omega \underset{\theta}{\oplus} \nu|\theta\right), \qquad \overset{\bar{\nu}}{D_\omega}[\theta]F(\omega|\theta):=F\left(\omega \underset{\theta}{\$} \nu|\theta\right)$$

such that

$$\varphi_\omega(x \underset{\theta}{(} \tau|\theta)=\varphi_\omega(x|\theta)\varphi_\omega(\tau|\theta), \qquad \varphi_\omega(x \underset{\theta}{'} \tau|\theta)=\varphi_\omega(x|\theta)\overline{\varphi}_\omega(\tau|\theta) \tag{5}$$

and

$$\varphi_{\omega \underset{\theta}{\oplus} \nu}(x|\theta)=\varphi_\omega(x|\theta)\varphi_\nu(x|\theta), \qquad \varphi_{\omega \underset{\theta}{\$} \nu}(x|\theta)=\varphi_\omega(x|\theta)\overline{\varphi}_\nu(x|\theta). \tag{6}$$

Here, symbols "$\underset{\theta}{(}$", "$\underset{\theta}{\oplus}$" and "$\underset{\theta}{\$}$", "$\underset{\theta}{'}$" denote the θ-parametrized quasi-sums and quasi-differences, respectively.

We will need in the following modulation operators:

$$\left(\overset{\nu}{M_x}[\theta]f\right)(x):=\varphi_\nu(x|\theta)f(x), \qquad \left(\overset{\tau}{M_\omega}[\theta]F\right)(\omega|\theta):=\overline{\varphi}_\omega(\tau|\theta)F(\omega|\theta).$$

From the GSOs definition it follows the following result (theorems about shifts and modulations). Shifts and modulations are connected as follows:

$$\widehat{T}_x^\tau[\theta]f(x):=f\left(x \underset{\theta}{\oplus} \tau\right) \quad \underset{CF}{\leftrightarrow} \quad \overset{\tau}{M_\omega}[\theta]F(\omega|\theta)=F(\omega|\theta)\overline{\varphi}_\omega(\tau|\theta),$$

$$\overset{\bar{\nu}}{D_\omega}[\theta]F(\omega|\theta)=F\left(\omega \underset{\theta}{(} \nu|\theta\right) \quad \underset{CF}{\leftrightarrow} \quad \overset{\nu}{M_x}[\theta]f(x|\theta)=f(x)\overline{\varphi}_\nu(x|\theta).$$

Two families of θ-parametrized GSOs $\left\{\widehat{T}_x^\upsilon\right\}_{\upsilon\in\Omega}=\left\{\widehat{T}_x^\upsilon[\theta]\right\}_{\upsilon\in\Omega}$ and $\left\{\widehat{D}_\omega^\nu\right\}_{\nu\in\Omega^*}=\left\{\widehat{D}_\omega^\nu[\theta]\right\}_{\nu\in\Omega^*}$ form two commutative hypergroups for every value of θ.

By definition, functions $\varphi_\omega(x|\theta)$ are eigenfunctions of GSOs. For this reason, we can call them *hypercharacters*. The idea of a hypercharacter on a hypergroup encompasses characters of locally compact and finite Abelian group and multiplication formulas for classical orthogonal polynomials. The theory of GSOs was initiated by Levitan (1949,1964) and (in the terminology of hypergroup) by Duncl (1966) and Jewett (1975). The class of commutative generalized translation hypergroups includes the class of locally compact and finite Abelian groups and semigroups. We will show that many well-known harmonic analysis theorems extend to the commutative hypergroups associated with arbitrary Fourier transforms.

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Generalized convolutions and correlations

It is well known that any stationary linear dynamic systems (LDS) are described by convolution integrals. Using the notion GSO, we can formally generalize the definitions convolution and correlation.

Definition 1. *The following functions*

$$y(x\,|\,\theta) := \left(h \underset{\theta}{\ddot{\mathrm{E}}} f\right)(x) = \int\limits_{\tau\in\Omega} h(\tau) f\left(x' \underset{\theta}{} \tau\right) d\mu(\tau), \tag{7}$$

$$Y(\omega\,|\,\theta) = \left(H \underset{\theta}{\heartsuit} F\right)(\omega) = \int\limits_{\nu\in\Omega} H(\nu) F\left(\omega \underset{\theta}{\$} \nu\right) d\mu(\nu), \tag{8}$$

and

$$\mathrm{cor}_{f,g}(\tau\,|\,\theta) = \left(f \underset{\theta}{\clubsuit} g\right)(\tau) = \int\limits_{x\in\Omega} f(x) \bar{g}\left(x' \underset{\theta}{} \tau\right) d\mu(x), \tag{9}$$

$$\mathrm{COR}_{F,G}(\nu\,|\,\theta) = \left(F \underset{\theta}{\spadesuit} G\right)(\nu) = \int\limits_{\omega\in\Omega^*} F(\omega) \bar{G}\left(\omega \underset{\theta}{\$} \nu\right) d\mu(\omega) \tag{10}$$

are called the θ-parametrized $\underset{\theta}{\ddot{\mathrm{E}}}$ - and $\underset{\theta}{\heartsuit}$ - convolutions and the cross $\underset{\theta}{\clubsuit}$ - and $\underset{\theta}{\spadesuit}$ - correlation functions, respectively, associated with a classical manyparameter Fourier transform $\mathrm{CF}[\theta]$. If $f = g$ and $F = G$ then cross correlation functions are called the $\underset{\theta}{\clubsuit}$ - and $\underset{\theta}{\spadesuit}$ -autocorrelation functions.

The spaces $\mathbf{ClSig}_0$ and $\mathbf{ClSp}_0$ equipped multiplications $\underset{\theta}{\ddot{\mathrm{E}}}$ - and $\underset{\theta}{\heartsuit}$ - convolutions form commutative Banach θ-parametrized signal and spectral convolution algebras $\left\langle\left\langle \mathbf{ClSig}_0, \underset{\theta}{\ddot{\mathrm{E}}} \right\rangle\right\rangle$ and $\left\langle\left\langle \mathbf{ClSp}_0, \underset{\theta}{\heartsuit} \right\rangle\right\rangle$, respectively. Generalized correlations and convolutions have many of the properties with group correlations and convolutions; many of them are catalogued in (Creutzburg *et al.,* 1992, 1994, 1998; Labunets *et al.,* 1976, 1980, 1982, 1993, 2000).

Theorem 1. *Generalized classical manyparametric Fourier transforms (3) and (4) map $\Diamond$-and $\heartsuit$-convolutions and $\clubsuit$- and $\spadesuit$-correlations into the products of spectra and signals, respectively,*

$$\mathrm{CF}[\theta]\left\{h \underset{\theta}{\ddot{\mathrm{E}}} f\right\} = \mathrm{CF}[\theta]\{h\} \cdot \mathrm{CF}[\theta]\{f\} = H(\omega\,|\,\theta) \cdot F(\omega\,|\,\theta),$$

$$\mathrm{CF}^{-1}[\theta]\left\{H \underset{\theta}{\heartsuit} F\right\} = \mathrm{CF}^{-1}[\theta]\{H\} \cdot \mathrm{CF}^{-1}[\theta]\{F\} = h(x) f(x),$$

$$\mathrm{CF}[\theta]\left\{f \underset{\theta}{\clubsuit} g\right\} = \mathrm{CF}[\theta]\{f\} \cdot \overline{\mathrm{CF}[\theta]\{g\}} = F(\omega\,|\,\theta) \cdot \bar{G}(\omega\,|\,\theta),$$

$$\mathrm{CF}^{-1}[\theta]\left\{F \underset{\theta}{\spadesuit} G\right\} = \mathrm{CF}^{-1}[\theta]\{F\} \cdot \overline{\mathrm{CF}^{-1}[\theta]\{G\}} = f(x) \bar{g}(x).$$

If classical generalized multiparametric Fourier transform $\mathrm{CF}[\theta]$ has fast classical or quantum realizations for all values of θ then this theorem gives us a fast procedure of calculating of generalized convolutions and correlations. When parameters are changed in some range, the type and form of transform $\mathrm{CF}[\theta]$ and GSOs are changed too. We can calculate convolutions and correlations for infinite number of transforms belonging to MPT. The sets all convolutions and correlations are called the *convolutiongram* and *correlationgram*. If in classical signal/image processing systems we take a look at single «photo» of convolution or correlation (for example, the ordinary convolution of an image in

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Fourier basis), then in multiparametric case we can look through two thrillers «CONVOLUTIONS OF SIGNALS» and «CORRELATIONS OF SIGNALS».

Fixing special values of θ we fixe special form of classical Fourier transform and, hence, the special familie of GSOs and, therefore, we can obtain special types of convolution and crosscorrelation: arithmetic, cyclic, dyadic, m-adic, etc. The output of a computation can be memorized to be used in the later numerical computations.

Generalized ambiguity functions and Wigner distributions

Along with the «time» and «frequency» domains we will work with «time-time» $\Omega\times\Omega$, «time-frequency» $\Omega\times\Omega^*$, «frequency-time» $\Omega^*\times\Omega$, and «frequency-frequency» $\Omega^*\times\Omega^*$ domains, and with four distributions, which are denoted by double letters $\mathrm{ff}(x,\upsilon)\in L(\Omega\times\Omega,\mathbf{A})$, $\mathrm{Ff}(\omega,\upsilon)\in L(\Omega^*\times\Omega,\mathbf{A})$, $\mathrm{fF}(x,\nu)\in L(\Omega\times\Omega^*,\mathbf{A})$ and $\mathrm{FF}(\omega,\nu)\in L(\Omega^*\times\Omega,\mathbf{A})$.

An important examples of time-frequency distribution are so-called *the symmetrical and asymmetrical Wigner-Ville distributions*. The Wigner asymmetric $\mathrm{wV}^a[f](x,\omega)$ and symmetric $\mathrm{wV}^s[f](x,\omega)$ distributions were introduced in 1932 by E. Wigner in the context of quantum mechanics, where he defined the probability function of the simultaneous values of the spatial coordinates and impulses.

Definition 2. *The ordinary symmetric and asymmetric Wigner-Ville distributions of two signals* f,g *are defined by*

$$\mathrm{wV}^s[f,g](x,\omega)=\underset{\omega\leftarrow\tau}{\mathrm{F}}\left\{f\left(x+\frac{\tau}{2}\right)\cdot\bar g\left(x-\frac{\tau}{2}\right)\right\}=\underset{\omega\leftarrow\tau}{\mathrm{F}}\left\{(f\cdot\bar g)^s(x,\tau)\right\},$$

$$\mathrm{wV}^a[f,g](x,\omega)=\underset{\omega\leftarrow\tau}{\mathrm{F}}\left\{f(x)\cdot\bar g(x-\tau)\right\}=\underset{\omega\leftarrow\tau}{\mathrm{F}}\left\{(f\cdot\bar g)^a(x,\tau)\right\},$$

where $(f\cdot\bar g)^s(x,\tau):=f\left(x+\frac{\tau}{2}\right)\cdot\bar g\left(x-\frac{\tau}{2}\right)$ *and* $(f\cdot\bar g)^a(x,\tau):=f(x)\cdot\bar g(x-\tau)$ *are so-called the symmetric and asymmetric products of two signals,* F *is the ordinary classical Fourier transform.*

Definition 3. *The ordinary symmetric and asymmetric Wigner-Ville distributions of two spectra* F,G *are defined by*

$$\mathrm{Wv}^s[F,G](\omega,x)=\underset{x\leftarrow\nu}{\mathrm{F}}^{-1}\left\{F\left(\omega+\frac{\nu}{2}\right)\cdot\bar G\left(\omega-\frac{\nu}{2}\right)\right\}=\underset{x\leftarrow\nu}{\mathrm{F}}^{-1}\left\{\left(F\cdot\bar G\right)^s(\omega,\nu)\right\},$$

$$\mathrm{Wv}^a[F,G](x,\omega)=\underset{x\leftarrow\nu}{\mathrm{F}}^{-1}\left\{F(\omega)\cdot\bar G(\omega-\nu)\right\}=\underset{x\leftarrow\nu}{\mathrm{F}}^{-1}\left\{\left(F\cdot\bar G\right)^a(\omega,\nu)\right\},$$

where $\left(F\cdot\bar G\right)^s(\omega,\nu):=F\left(\omega+\frac{\nu}{2}\right)\cdot\bar G\left(\omega-\frac{\nu}{2}\right)$ *and* $\left(F\cdot\bar G\right)^a(\omega,\nu):=F(\omega)\cdot\bar G(\omega-\nu)$ *are so-called the symmetric and asymmetric products of two spectra.*

Wigner's idea was introduced in signal analysis in 1948 by J. Ville, but it did not receive much attention there until 1951 when P. Woodward reformulated it in the context of radar theory. Woodward proposed treating the question of radar signal ambiguity as a part of the question of target resolution. For that, he introduced functions $\mathrm{Aw}^s[f](\nu,\tau)$ and $\mathrm{Aw}^a[f](\nu,\tau)$ that described the correlation between a radar signal and its Doppler-shifted and time-translated version.

Definition 4. *The ordinary symmetric and asymmetric cross-ambiguity functions of two signals* f,g *are defined by*

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$$\mathrm{Aw}^{s}[f,g](\nu,\tau)=\underset{\nu\leftarrow x}{\mathrm{F}}\left\{f\left(x+\frac{\tau}{2}\right)\cdot\overline{g}\left(x-\frac{\tau}{2}\right)\right\}=\underset{\nu\leftarrow x}{\mathrm{F}}\left\{\left(f\cdot\overline{g}\right)^{s}(x,\tau)\right\},$$

$$\mathrm{Aw}^{a}[f,g](\nu,\tau)=\underset{\nu\leftarrow x}{\mathrm{F}}\left\{f(x)\cdot\overline{g}(x-\tau)\right\}=\underset{\nu\leftarrow x}{\mathrm{F}}\left\{\left(f\cdot\overline{g}\right)^{a}(x,\tau)\right\},$$

where $\left(f\cdot\overline{g}\right)^{s}(x,\tau):=f\left(x+\frac{\tau}{2}\right)\cdot\overline{g}\left(x-\frac{\tau}{2}\right)$ *and* $\left(f\cdot\overline{g}\right)^{a}(x,\tau):=f(x)\cdot\overline{g}(x-\tau)$ *are so-called the temporal generalized local cross-correlation functions.*

Definition 5. *The ordinary symmetric and asymmetric cross-ambiguity functions of two spectra* F,G *are defined by*

$$\mathrm{aW}^{s}\left[F,G\right](\tau,\nu)=\underset{\tau\leftarrow\omega}{\mathrm{F}}^{-1}\left\{F\left(\omega+\frac{\nu}{2}\right)\cdot\overline{G}\left(\omega-\frac{\nu}{2}\right)\right\}=\underset{\tau\leftarrow\omega}{\mathrm{F}}^{-1}\left\{\left(F\cdot\overline{G}\right)^{s}(\omega,\nu)\right\},$$

$$\mathrm{aW}^{a}\left[F,G\right](\tau,\nu)=\underset{\tau\leftarrow\omega}{\mathrm{F}}^{-1}\left\{F(\omega)\cdot\overline{G}(\omega-\nu)\right\}=\underset{\tau\leftarrow\omega}{\mathrm{F}}^{-1}\left\{\left(F\cdot\overline{G}\right)^{a}(\omega,\nu)\right\},$$

where $\left(F\cdot\overline{G}\right)^{s}(\omega,\nu):=F\left(\omega+\frac{\nu}{2}\right)\cdot\overline{G}\left(\omega-\frac{\nu}{2}\right)$ *and* $\left(F\cdot\overline{G}\right)^{a}(\omega,\nu):=F(\omega)\cdot\overline{G}(\omega-\nu)$ *are so-called the spectral generalized local cross-correlation functions.*

Using the notion GSO, we can formally generalize the notions of ambiguity functions and Wigner-Ville distributions.

Definition 6. *The generalized symmetric and asymmetric Wigner-Ville distributions of two signals* f,g *are defined by*

$$\mathrm{wV}^{s}\left[f,g\right](x,\omega)=\underset{\omega\leftarrow\tau}{\mathrm{CF}}[\theta]\left\{f\left(x\underset{\theta}{(}\frac{\tau}{2}\right)\cdot\overline{g}\left(x\underset{\theta}{'}\frac{\tau}{2}\right)\right\}=\underset{\omega\leftarrow\tau}{\mathrm{CF}}[\theta]\left\{\left(f\cdot\overline{g}\right)^{s}(x,\tau)\right\},$$

$$\mathrm{wV}^{a}\left[f,g\right](x,\omega)=\underset{\omega\leftarrow\tau}{\mathrm{CF}}[\theta]\left\{f\left(x\right)\cdot\overline{g}\left(x\underset{\theta}{'}\tau\right)\right\}=\underset{\omega\leftarrow\tau}{\mathrm{CF}}[\theta]\left\{\left(f\cdot\overline{g}\right)^{s}(x,\tau)\right\}.$$

Definition 7. *The generalized symmetric and asymmetric Wigner-Ville distributions of two spectra* F,G *are defined by*

$$\mathrm{Wv}^{s}\left[F,G\right](\omega,x)=\underset{x\leftarrow\nu}{\mathrm{CF}}^{-1}[\theta]\left\{F\left(\omega\underset{\theta}{\oplus}\frac{\nu}{2}\right)\cdot\overline{G}\left(\omega\underset{\theta}{\$}\frac{\nu}{2}\right)\right\}=\underset{x\leftarrow\nu}{\mathrm{CF}}^{-1}[\theta]\left\{\left(F\cdot\overline{G}\right)^{s}(\omega,\nu)\right\},$$

$$\mathrm{Wv}^{a}\left[F,G\right](\omega,x)=\underset{x\leftarrow\nu}{\mathrm{CF}}^{-1}[\theta]\left\{F(\omega)\cdot\overline{G}\left(\omega\underset{\theta}{\$}\nu\right)\right\}=\underset{x\leftarrow\nu}{\mathrm{CF}}^{-1}[\theta]\left\{F\cdot\overline{G}^{s}(\omega,\nu)\right\}.$$

Definition 8. *The generalized symmetric and asymmetric cross-ambiguity functions of two signals* f,g *are defined by*

$$\mathrm{Aw}^{s}\left[f,g\right](\nu,\tau)=\underset{\nu\leftarrow x}{\mathrm{CF}}[\theta]\left\{f\left(x\underset{\theta}{(}\frac{\tau}{2}\right)\cdot\overline{g}\left(x\underset{\theta}{'}\frac{\tau}{2}\right)\right\}=\underset{\nu\leftarrow x}{\mathrm{CF}}[\theta]\left\{\left(f\cdot\overline{g}\right)^{s}(x,\tau)\right\},$$

$$\mathrm{Aw}^{a}\left[f,g\right](\nu,\tau)=\underset{\nu\leftarrow x}{\mathrm{CF}}[\theta]\left\{f(x)\cdot\overline{g}\left(x\underset{\theta}{'}\tau\right)\right\}=\underset{\nu\leftarrow x}{\mathrm{CF}}[\theta]\left\{\left(f\cdot\overline{g}\right)^{a}(x,\tau)\right\},$$

where $\left(f\overline{g}\right)^{s}(x,\tau):=f\left(x\underset{\theta}{(}\frac{\tau}{2}\right)\overline{g}\left(x\underset{\theta}{'}\frac{\tau}{2}\right)$ *and* $\left(f\cdot g\right)^{a}(x,\tau):=f\left(x\right)\cdot\overline{g}\left(x\underset{\theta}{'}\tau\right)$ *are so-called temporal generalized local cross-correlation functions.*

Definition 9. *The generalized symmetric and asymmetric cross-ambiguity functions of two spectra* F,G *are defined by*

$$\mathrm{aW}^{s}\left[F,G\right](\tau,\nu)=\underset{\tau\leftarrow\omega}{\mathrm{CF}}^{-1}[\theta]\left\{F\left(\omega\underset{\theta}{\oplus}\frac{\nu}{2}\right)\overline{G}\left(\omega\underset{\theta}{\$}\frac{\nu}{2}\right)\right\}=\underset{\tau\leftarrow\omega}{\mathrm{CF}}^{-1}[\theta]\left\{F\overline{G}^{s}(\omega,\nu)\right\},$$

$$\mathrm{aW}^{a}\left[F,G\right](\tau,\nu)=\underset{\tau\leftarrow\omega}{\mathrm{CF}}^{-1}[\theta]\left\{F(\omega)\cdot\overline{G}\left(\omega\underset{\theta}{\$}\nu\right)\right\}=\underset{\tau\leftarrow\omega}{\mathrm{CF}}^{-1}[\theta]\left\{\left(F\cdot\overline{G}\right)^{a}(\omega,\nu)\right\},$$

Электронный архив УГЛТУ

where $\left(F\cdot\overline{G}\right)^{s}(\omega,\nu):=F\left(\omega\underset{\theta}{\oplus}\frac{\nu}{2}\right)\cdot\overline{G}\left(\omega\underset{\theta}{\$}\frac{\nu}{2}\right)$ *and* $\left(F\cdot\overline{G}\right)^{a}(\omega,\nu):=F(\omega)\cdot\overline{G}\left(\omega\underset{\theta}{\$}\nu\right)$ *are so-called the spectral generalized local cross-correlation functions.*

We see that the Wigner-Ville distributions are the 2D symplectic Fourier transform of $\mathrm{aW}^{s}\left[f,g\right](\tau,\nu)$ and $\mathrm{aW}^{a}\left[f,g\right](\tau,\nu)$, respectively:

$$\mathrm{wV}\left[f,g\right](x,\omega)=\underset{\omega\leftarrow\tau}{\mathcal{CF}}[\theta]\underset{x\leftarrow\nu}{\mathcal{CF}}^{-1}[\theta]\left\{\mathrm{aW}\left[f,g\right](\tau,\nu)\right\}, \tag{11}$$

$$\mathrm{Wv}\left[F,G\right](\omega,x)=\underset{x\leftarrow\nu}{\mathcal{CF}}^{-1}[\theta]\underset{\omega\leftarrow\tau}{\mathcal{CF}}[\theta]\left\{\mathrm{aW}\left[F,G\right](\tau,\nu)\right\}. \tag{12}$$

The 2D symplectic Fourier transform in (11) and (12) can be also viewed as performing two subsequently 1D transforms with respect to τ and ν.

Conclusion

In this paper we developed generalized nonharmonic analysis of signals and images on commutative hypergroups, associated with arbitrary unitary (orthogonal) transforms. We introduced generalized convolutions, correlations, Wigner-Ville distributions, and ambiguity functions. All theorems and properties of ordinary classical Fourier harmonic analysis are transferred on nonharmonic analysis Fourier on arbitrary Abelian hypergroups.

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Рецензент статьи: кандидат технических наук, доцент Института радиоэлектроники и информационных технологий Уральского федерального университета С.М. Зраенко.