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GENERALIZED CLASSICAL AND QUANTUM SIGNAL THEORIES ON HYPERGROUPS. PART 2. QUANTUM SIGNAL THEORY



Introduction

Quantum signal theory is a term referring to a collection of ideas and partial results, loosely held together, assuming that there are deep connections between the worlds of quantum physics and classical signal/system theory, and that one should try to discover and develop these connections. The general topic of this part of our program is the following idea. If some algebraic structures arise together in quantum theory and classical signal/system theory in the same context, then one should try to make sense of this for more generalized algebraic structures. Here, the point is not to try to develop alternative theories as substitute models for quantum physics and signal/system theory, but rather to develop a «β-version» of a *unified scheme of general classical and quantum signal/system theory*.

It is known (Creutzburg *at al.,* 1992, 1994) that general building elements of the *Classical and Quantum Signal/System Theories* (**Cl-SST** and **Qu-SST**) are the following:

1) Abelian group of real numbers **AR**,
2) classical Fourier transform $\mathcal{F}$, and
3) complex field **C**.

This means that these theories are associated with the triple $\langle\langle \mathbf{AR}, \mathcal{F}, \mathbf{C}\rangle\rangle$. We can write

$$\mathbf{Cl\text{-}SST} = f_{cl}\left(\langle\langle \mathbf{AR}, \mathcal{F}, \mathbf{C}\rangle\rangle\right), \quad \mathbf{Qu\text{-}SST} = f_{qu}\left(\langle\langle \mathbf{AR}, \mathcal{F}, \mathbf{C}\rangle\rangle\right)$$

for any correspondences f_{cl} and f_{qu}, respectively. These correspondences mean that triple $\langle\langle \mathbf{AR}, \mathcal{F}, \mathbf{C}\rangle\rangle$ determines ordinary theories **Cl-SST** and **Qu-SST**.

We develop a new unified approach to the *Generalized Classical (see Part 1) and Quantum Signal/System Theories* (**GCl-SST** and **GQu-SST**). They are based not on the triple $\langle\langle \mathbf{AR}, \mathcal{F}, \mathbf{C}\rangle\rangle$, but rather on other Abelian groups and hypergroups, on a large class of orthogonal, unitary, multiorthounitary multiparametric transforms (instead of the classical Fourier transform), and involves other fields, rings and algebras (triplet color algebra, multiplet multicolor algebra, hypercomplex commutative algebras, Clifford algebras). In our approach Generalized Classical and Quantum Signal/System Theories are two functions (correspondences) of a new triple:

$$\mathbf{GCl\text{-}SST} = f_{cl}\left(\left\langle\left\langle \mathbf{HG}, \mathcal{F}, \mathbf{A}\right\rangle\right\rangle\right), \quad \mathbf{GQu\text{-}SST} = f_{qu}\left(\left\langle\left\langle \mathbf{HG}, \mathcal{F}, \mathbf{A}\right\rangle\right\rangle\right),$$

where **HG** is a hypergroup, $\mathcal{F}$ is a unitary transform, and $\mathbf{A}$ is an algebra. When the triple $\left\langle\left\langle \mathrm{HG}, \mathcal{F}, \mathbf{A}\right\rangle\right\rangle$ is changed the theories **GCl-SST** and **GQu-SST** are changed, too. For example,

- If $\mathcal{F}$ is the classical Fourier transform, **HG** is the group of real numbers **AR** and $\mathbf{A}$ is the complex field **C**, then $\left\langle\left\langle \mathbf{AR}, \mathcal{F}, \mathbf{C}\right\rangle\right\rangle$ describes free quantum particle.
- If $\mathcal{F}$ is the classical Walsh transform (CWT), **HG** is an abelian dyadic group $\mathbf{Z}_2^n$ and $\mathbf{A}$ is the complex field **C**, then $\left\langle\left\langle \mathbf{Z}_2^n, \mathrm{CWT}, \mathbf{A}\right\rangle\right\rangle$ describes n-digital quantum register.
- If $\mathcal{F}$ is the classical Vilenkin transform (CVT), **HG** is an abelian m-adic group $\mathbf{Z}_m^n$ and $\mathbf{A}$ is the complex field **C**, then $\left\langle\left\langle \mathbf{Z}_m^n, \mathbf{CWT}, \mathbf{C}\right\rangle\right\rangle$ describes n-digital quantum m-adic register.

Different triples generate a wide class of classical and quantum signal theories. Using multiparametric transforms we construct so-called *Multiparametric Generalized Classical and Quantum Signal Theories*

$$\mathbf{MP\text{-}GCl\text{-}SST} = f_{cl}\left(\left\langle\left\langle \mathbf{HG}, \mathcal{CF}[\theta], \mathbf{A}\right\rangle\right\rangle\right), \quad \mathbf{MP\text{-}GQu\text{-}SST} = f_{qu}\left(\left\langle\left\langle \mathbf{HG}, \mathcal{CF}[\theta], A\right\rangle\right\rangle\right).$$

We develop two topics (Multiparametric Generalized Classical and Quantum Signal/System Theories) in sequence and show their inter- and cross-relation. We study classical and quantum generalized convolution hypergroup algebras of classical signals and Hermitian operators (quantum signals). One of the main purposes of this work is to demonstrate parallelism between the generalized classical hyperharmonic analysis and the generalized quantum hyperharmonic analysis.

Basic definitions

Quantum signal theory is a term referring to a collection of ideas and partial results, loosely held together, suggesting that there are deep connections between the worlds of quantum physics and classical signal/system theory, and that one should try to discover and develop these connections.

The basic objects of the quantum signal theory are related not to *functions* but to *Hermitean operators* $\hat{f}$, $\hat{F}$ associated with classical signals f and spectra F by so-called Weyl quantization rule (Weyl, 1931):

$$\mathbf{WQ}: f \to \mathrm{Aw}\left[f\right] \to \hat{f}, \ \mathbf{WQ}: F \to \mathrm{aW}\left[F\right] \to \hat{F}.$$

(There are the *Schwinger quantization rule* using Wigner-Ville distributions). To obtain quantum representation of signals and spectra we have first to represent this object classically by time-frequency distributions $\mathrm{Aw}[f](\nu,\tau)$ and $\mathrm{aW}[F](\tau,\nu)$ and then quantize those representations using so-called generalized quantum Fourier transforms $\hat{f} = \mathcal{QF}\left\{\mathrm{Aw}[f](\nu,\tau)\right\}$ and $\hat{F} = \mathcal{QF}\left\{\mathrm{aW}[F](\tau,\nu)\right\}$. We see that the generalized Weyl quantization is a bilinear operator mapping every basic object of the classical signal theory to basic object of the quantum signal theory. This operator is a composition of the generalized Woodward-Gabor transform and generalized quantum Fourier transform both associated with classical generalized Fourier transform $\mathcal{CF}$. The functions $\mathrm{Aw}[f](\nu,\tau)$, $\mathrm{aW}[F](\tau,\nu)$ (or

$\mathrm{wV}[f](x,\omega)$, $\mathrm{Wv}[F](\omega,x)$) are called the *symbols* (a symbol is not a kernel) of the quantum signal $\hat{f}$ and the quantum spectra $\hat{F}$, respectively, and are denoted as $\mathbf{Aw}[f](\nu,\tau) := sym\{\hat{f}\}$, and $\mathrm{aW}[F](\tau,\nu) := sym\{\hat{F}\}$. Vice versa, a quantum signal $\hat{f}$ and quantum spectra $\hat{F}$ are called the *operators associated with a classical signal* f and classical spectrum by symbols $\mathrm{Aw}[f]$ and $\mathrm{aW}[F]$, respectively, and they are denoted as $\hat{f} := Op\{\mathrm{Aw}[f]\}$, and $\hat{F} := Op\{\mathrm{aW}[F]\}$.

Quantization rules

Classical Weyl quantization. In 1931 H. Weyl proposed to modify the classical ordinary Fourier transform formula by changing its *complex-valued harmonics* into an *operator-valued harmonics*. He used the following operator-valued harmonics:

$$\left\{\hat{\mathrm{E}}_x^{[\nu,\tau]} := e^{i\left(\nu\hat{\mathcal{M}}_x+\tau\hat{\mathcal{D}}_x\right)} = e^{i\nu\tau/2}e^{i\nu\hat{\mathcal{M}}_x}e^{i\tau\hat{\mathcal{D}}_x}\right\} \tag{1}$$

associated with classical ordinary Fourier transform, where multiplication $\hat{\mathcal{M}}_x$ and differential $\hat{\mathcal{D}}_x$ operators are given by $\hat{\mathcal{M}}_x f(x) := xf(x)$, $\hat{\mathcal{D}}_x f(x) := -\dfrac{df(x)}{dx}$. Using these operator-valued harmonics, H. Weyl wrote any quantum signal $\hat{f}$ as

$$\hat{f} := \mathrm{QF}_x\{\mathrm{Aw}[f]\} = \int\limits_{\nu\in\Omega^*}\int\limits_{\tau\in\Omega} \mathrm{Aw}[f](\nu,\tau)e^{i\left[\nu\hat{\mathcal{M}}_x+\tau\hat{\mathcal{D}}_x\right]}d\,\mu(\nu)d\,\mu(\tau), \tag{2}$$

where

$$\mathrm{Aw}[f](\nu,\tau) = \mathrm{QF}_x^{-1}\{\hat{f}\} = Sym\{\hat{f}\} = \mathbf{Tr}\left[\hat{f}\cdot e^{-i\left[\nu\hat{\mathcal{M}}_x+\tau\hat{\mathcal{D}}_x\right]}\right]. \tag{3}$$

Transformations (2) and (3) are called the direct and inverse *quantum ordinary Fourier transforms*.

It is well known that for classical shift we have $\hat{T}_x^\tau f(x) := f(x+\tau) = e^{\tau\hat{\mathcal{D}}_x}f(x)$. This expression represents the decomposition of ordinary finite shift into a series of powers of differential operator $\dfrac{d}{dx}$ and is called the *infinitesimal representation* of translation shift. Analogously, we can obtain $\hat{D}_\omega^\nu F(\omega) = F(\omega+\nu) = e^{\nu\hat{\mathcal{D}}_\omega}F(\omega)$, where $\hat{\mathcal{D}}_\omega = \dfrac{d}{d\omega}$. Hence, $\hat{T}_x^\tau = e^{\tau\hat{\mathcal{D}}_x}$, $\hat{D}_\omega^\nu = e^{\nu\hat{\mathcal{D}}_\omega}$. For this reason, we can write operator-valued harmonics (in "time" and "frequency" domains) in the following forms

$$\mathrm{QF}_x = e^{i\left[\nu\hat{\mathcal{M}}_x+\tau\hat{\mathcal{D}}_x\right]} = e^{i\nu\tau/2}\hat{M}_x^\nu\hat{T}_x^\tau \text{ and } \mathrm{QF}_\omega = e^{i\left[\tau\hat{\mathcal{M}}_\omega+\nu\hat{\mathcal{D}}_\omega\right]} = e^{i\nu\tau/2}\hat{M}_\omega^\tau\hat{D}_\omega^\nu. \tag{4}$$

We shall use these expressions for design of generalized quantum Fourier transforms, associated with generalized classical Fourier transforms (including MPTs).

Generalized Weyl quantization. Let us construct generalized operator-valued hyperharmonics associated with a basis $\{\varphi_\omega(x)\}_{\omega\in\Omega^*}$. This basic can be a multiparametric basis $\{\varphi_\omega(x)\}_{\omega\in\Omega^*} = \{\varphi_\omega(x|\theta)\}_{\omega\in\Omega^*}$ and can generates a multiparameter classical Fourier transform $\mathcal{CF} = \mathcal{CF}[\theta]$. All generalized shift operators associated with $\mathcal{CF}[\theta]$ have the following infinitesimal representation:

$$\overset{\tau}{T}_x[\theta] = \varphi_{\mathcal{D}_x}(\tau\,|\,\theta), \quad \overset{\mathrm{v}}{D}_\omega[\theta] = \varphi_\nu\left(\mathcal{D}_\omega\,|\,\theta\right),$$

and are called the *operator-valued hyperharmonics* associated with a manyparameter orthogonal basis $\{\varphi_\omega(x\,|\,\theta)\}_{\omega\in\Omega^*}$, where

$$\mathcal{D}_x[\theta]\varphi_\omega(x\,|\,\theta)=\omega\varphi_\omega(x\,|\,\theta),\ \mathcal{D}_\omega[\theta]\varphi_\omega(x\,|\,\theta)=x\varphi_\omega(x\,|\,\theta).$$

Using operator-valued hyperharmonics we can construct generalized Heisenberg-Weyl operators (according to (1)) and quantum hyperharmonic analysis of quantum signals and spectra. Symmetric Heisenberg-Weyl operators we define as:

$$\mathrm{E}_x^{[\nu,\tau]}[\theta]=\varphi_\nu^{1/2}(\tau\,|\,\theta)\overset{\nu}{M}_x[\theta]\overset{\tau}{T}_x[\theta]=\varphi_\nu^{1/2}(\tau\,|\,\theta)\varphi_\nu\left(\mathcal{M}_x\,|\,\theta\right)\varphi_{\mathcal{D}_x}(\tau\,|\,\theta),\quad \nu\in\Omega^*,\tau\in\Omega, \tag{5}$$

$$\mathrm{E}_\omega^{[\tau,\nu]}[\theta]=\varphi_\nu^{1/2}(\tau\,|\,\theta)\overset{\tau}{M}_\omega[\theta]\overset{\nu}{D}_\omega[\theta]=\varphi_\nu^{1/2}(\tau\,|\,\theta)\varphi_{\mathcal{M}_\omega}\left(\tau\,|\,\theta\right)\varphi_\omega(\mathcal{D}_\nu\,|\,\theta),\quad \nu\in\Omega^*,\tau\in\Omega. \tag{6}$$

These operators satisfy the following composition laws and the «commutation» relations:

$$\mathrm{E}_x^{[\nu,\tau]}[\theta]\cdot\mathrm{E}_x^{[\nu',\tau']}[\theta]=\bar{\varphi}_\nu^{1/2}(\tau'\,|\,\theta)\cdot\varphi_{\nu'}^{1/2}(\tau\,|\,\theta)\cdot\mathrm{E}_x^{[\nu+\nu',\tau+\tau']}[\theta],$$

$$\mathrm{E}_x^{[\nu,\tau]}[\theta]\cdot\mathrm{E}_x^{[\nu',\tau']}[\theta]=\bar{\phi}_\nu^{1/2}(\tau'\,|\,\theta)\cdot\phi_{\nu'}^{1/2}(\tau\,|\,\theta)\cdot\mathrm{E}_x^{[\nu',\tau']}[\theta]\cdot\mathrm{E}_x^{[\nu,\tau]}[\theta],$$

and

$$\mathrm{E}_\omega^{[\tau,\nu]}[\theta]\cdot\mathrm{E}_\omega^{[\tau',\nu']}[\theta]=\bar{\varphi}_\nu^{1/2}(\tau'\,|\,\theta)\cdot\varphi_{\nu'}^{1/2}(\tau\,|\,\theta)\cdot\mathrm{E}_\omega^{[\tau+\tau',\nu+\nu']}[\theta],$$

$$\mathrm{E}_\omega^{[\tau,\nu]}[\theta]\cdot\mathrm{E}_\omega^{[\tau',\nu']}[\theta]=\bar{\varphi}_\nu^{1/2}(\tau'\,|\,\theta)\varphi_{\nu'}^{1/2}(\tau\,|\,\theta)\cdot\mathrm{E}_\omega^{[\tau',\nu']}[\theta]\cdot\mathrm{E}_\omega^{[\tau,\nu]}[\theta].$$

They form two Heisenberg-Weyl hypergroups associated with the generalized classical Fourier transform $\mathcal{CF}$. It is easy to check that

$$\mathbf{Tr}\left[\mathrm{E}_x^{[\nu,\tau]}[\theta]\cdot\left(\mathrm{E}_x^{[\nu',\tau']}[\theta]\right)^+\right]=\delta(\nu-\nu')\delta(\tau-\tau'), \tag{7}$$

$$\mathbf{Tr}\left[\mathrm{E}_\omega^{[\tau,\nu]}[\theta]\cdot\left(\mathrm{E}_\omega^{[\tau',\nu']}[\theta]\right)^+\right]=\delta(\nu-\nu')\delta(\tau-\tau'). \tag{8}$$

For this reason, for each value of parameter θ we can construct any quantum signal $f[\theta]$ and quantum spectrum $F[\theta]$, that can be written as follows:

$$f[\theta]=\mathrm{QF}_x[\theta]\left\{\mathrm{Aw}\left[f\,|\,\theta\right]\right\}=Op_\theta\left\{\mathrm{Aw}\left[f\,|\,\theta\right]\right\}=\int\limits_{\nu\in\Omega^*}\int\limits_{\tau\in\Omega}\mathrm{Aw}[f\,|\,\theta](\nu,\tau)\cdot\mathrm{E}_x^{[\nu,\tau]}[\theta]\cdot d\,\mu(\nu)d\,\mu(\tau), \tag{9}$$

$$F[\theta]=\mathrm{QF}_\omega[\theta]\left\{\mathrm{aW}\left[F\,|\,\theta\right]\right\}=Op_\theta\left\{\mathrm{aW}\left[F\,|\,\theta\right]\right\}=\int\limits_{\tau\in\Omega}\int\limits_{\nu\in\Omega^*}\mathrm{aW}[F\,|\,\theta](\tau,\nu)\cdot\mathrm{E}_\omega^{[\tau,\nu]}[\theta]\cdot d\,\mu(\tau)d\,\mu(\nu). \tag{10}$$

Using (7) and (8), one can invert (9) and (10) as follows:

$$\mathrm{Aw}[f\,|\,\theta](\nu,\tau)=\mathrm{QF}_x^{-1}[\theta]\left\{f[\theta]\right\}=Sym_\theta\left\{f[\theta]\right\}=\mathbf{Tr}\left[f[\theta]\cdot\left(\mathrm{E}_x^{[\nu,\tau]}[\theta]\right)^+\right], \tag{11}$$

$$\mathrm{aW}[F\,|\,\theta](\nu,\tau)=\mathrm{QF}_\omega^{-1}[\theta]\left\{F[\theta]\right\}=Sym_\theta\left\{F[\theta]\right\}=\mathbf{Tr}\left[F[\theta]\cdot\left(\mathrm{E}_\omega^{[\tau,\nu]}[\theta]\right)^+\right]. \tag{12}$$

Transformations (9-10) and (11-12) are called the *generalized direct and inverse manyparameter quantum Fourier transforms* associated with the multiparameter classical Fourier transform $\mathcal{CF}=\mathcal{CF}[\theta]$.

Quantum Fourier transform associated with classical DFT on the cyclic group $\mathbf{Z}_p$

Quantum DFT on the cyclic group $\mathbf{Z}_p$. As example, we investigate quantum discrete Fourier transform (DFT), associated with the classical DFT on a cyclic group $\mathbf{Z}_p$. Let $\Omega = \Omega^* = \mathbf{Z}_p$ be the finite cyclic group, where p is a prime integer. We use common formula

$$\hat{\mathrm{E}}_x^{[\nu,\tau]} = e^{i\nu\tau/2} e^{i\nu\hat{\mathcal{M}}_x} e^{i\tau\hat{\mathcal{D}}_x} = e^{i\nu\tau/2} \cdot M_x^{\nu} \cdot T_x^{\tau}.$$

For this reason, the map

$$\hat{f} = \mathrm{QF}_x\{\mathrm{Aw}[f]\} = Op\{\mathrm{Aw}[f]\} = \sum_{\nu\in\mathbf{Z}/p} \sum_{\tau\in\mathbf{Z}/p} \mathrm{Aw}[f](\nu,\tau) e^{i\nu\tau/2} M_x^{\nu} T_x^{\tau} =$$

$$= \sum_{\nu\in Z/p} \sum_{\tau\in Z/p} \left(\mathrm{AW}[f](\nu,\tau) \cdot e^{\frac{\nu\tau}{2}} \cdot \begin{bmatrix} 1 & & & & \\ & \varepsilon^1 & & & \\ & & \varepsilon^2 & & \\ & & & \ddots & \\ & & & & \varepsilon^{p-1} \end{bmatrix}^{\nu} \cdot \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}^{\tau} \right) \tag{13}$$

is the *quantum* DFT associated on the cyclic group $\mathbf{Z}_p$, where

$$M_x = \begin{bmatrix} 1 & & & & \\ & \varepsilon^1 & & & \\ & & \varepsilon^2 & & \\ & & & \ddots & \\ & & & & \varepsilon^{p-1} \end{bmatrix}, \quad T_x = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}.$$

Heisenberg-Weyl groups over ring $\mathbf{Z}_p$. Let us consider Heisenberg-Weyl group with elements of finite commutative ring over $\mathbf{Z}_p$:

$$\mathrm{HW} = \mathrm{HW}(\mathbf{Z}_p,\mathbf{Z}_p,\mathbf{Z}_p) := \left\{ g(t,\omega,c) = \begin{bmatrix} 1 & \omega & c \\ & 1 & t \\ & & 1 \end{bmatrix} \middle| t,\omega,c \in \mathbf{Z}_p \right\}$$

Consisting of upper triangular (3×3)-matrices $g(t,\omega,c)$ with the following multiplication rule: $g(t_1,\omega_1,c_1)\cdot g(t_2,\omega_2,c_2) = g(t_1+t_2,\omega_1+\omega_2,c_1+c_2+t_1\omega_2)$. Every element $g(t,\omega,c)$ has the unique representation of the following form: $g(t,\omega,c) = T^t \Omega^\omega C^c$, where

$$C^c := g(0,0,c) = \begin{bmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{bmatrix}, \quad T^t := g(t,0,0) = \begin{bmatrix} 1 & & \\ & 1 & t \\ & & 1 \end{bmatrix}, \quad \Omega^\omega := g(0,\omega,0) = \begin{bmatrix} 1 & \omega & \\ & 1 & \\ & & 1 \end{bmatrix}.$$

It is known that group $\mathrm{HW}(\mathbf{Z}_p,\mathbf{Z}_p,\mathbf{Z}_p)$ has p^2 1-D irreducible representations and $p-1$ p-D representations. Fourier transform of an arbitrary signal $\mathrm{fF}(t,\omega,c): \mathrm{HW}(\mathbf{Z}_p,\mathbf{Z}_p,\mathbf{Z}_p) \to \mathbf{C}$, defined on the Heisenberg group and with values in the complex field $\mathbf{C}$, has the following form:

- for scalar -valued spectra (p^2 1-D irreducible representations):

$$f_{1\times 1}(\alpha_1,\alpha_2)=\sum_{t,\omega,c}\sum_{\in}\sum_{Z_p} f(t,\omega,c)\,\varepsilon_p^{\alpha_1 t}\varepsilon_p^{\alpha_2\omega}, \tag{14}$$

- for matrix-valued spectra ($p-1$ p -D representations):

$$f_{p\times p}(\alpha_3)=\sum_{t,\omega,c}\sum_{\in}\sum_{\mathbf{Z}_p} f(t,\omega,c)\,\varepsilon_p^{\alpha_3 c}\begin{bmatrix}1 & & & & \\ & \varepsilon^1 & & & \\ & & \varepsilon^2 & & \\ & & & \ddots & \\ & & & & \varepsilon^{p-1}\end{bmatrix}^{\alpha_3\omega}\begin{bmatrix}0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 1 & & & & 0\end{bmatrix}^{t}, \tag{15}$$

where $\varepsilon_p=\sqrt[p]{1},\ \ \alpha_3=1,2,...,p-1$.

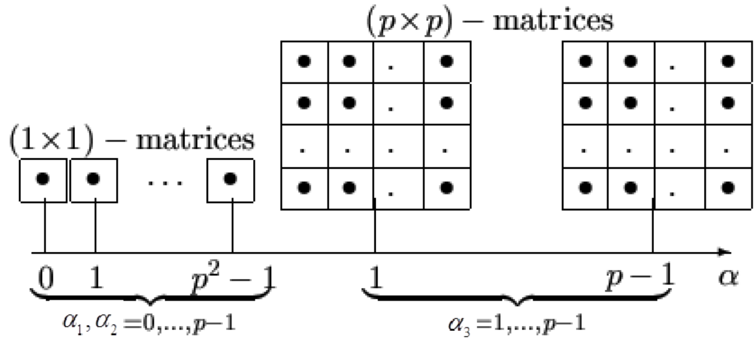

Fig. 1. Matrix-valued spectrum.

Hence, an arbitrary signal $f\mathrm{F}(t,\omega,c)$ defined on the Heisenberg group and with values in the complex field **C** has p^2 scalar-valued and $p-1$ $(p\times p)$-matrix-valued spectral components (see Fig. 1). With quantum mechanics point of view, Exp. (15) represents $p-1$ different quantum Fourier transforms (for every $\alpha_3=1,2,\dots,p-1$). The family of all 1D representations (14) corresponds to the classical mechanics ((α_1,α_2)-plane) and various representations (15) (where $\alpha_3=1,2,\dots,p-1$) lead to different quantum signals:

$$f_1=\mathbf{F}_{p\times p}(1),\ f_2=\mathbf{F}_{p\times p}(2),\ f_3=\mathbf{F}_{p\times p}(3),\dots,f_{p-1}=\mathbf{F}_{p\times p}(p-1).$$

These quantum words correspond to different quantum descriptions with different Planck's constant $\hbar=\alpha_3$.

Classical realization of quantum Fourier transform. We construct transform matrix QF_x for the quantum DFT (14-15) in the following way: the matrix rows will be enumerated with

$$m=(\alpha_1,\alpha_2,\alpha_3,i,k)=\begin{cases}\alpha_2 p+\alpha_1, & \text{for } 0\le m\le p^2-1,\\ p^2\alpha_3+ip+k, & \text{for } p^2\le m\le p^3-1,\end{cases}$$

where $\alpha_1,\alpha_2\in\mathbf{Z}_p, f_{p\times p}(\alpha_3)=[f_{ik}(\alpha_3)]_{i,k=0}^{p-1}$. The column with the number $n=t+\omega p+tp^2$ corresponds to the element $\overset{t}{T}\,\overset{\omega}{\Omega}\,\overset{c}{C}$, $t,\omega,c=0,1,\dots,p-1$. Then

$$\begin{aligned}\mathrm{QF}_x&=[I_{p^3-p^2}\oplus(I_p\otimes\mathrm{F}_p)][\mathrm{F}_p\otimes\mathrm{F}_p\otimes I_p]=\\ &=[I_{p^3-p^2}\oplus(I_p\otimes\mathrm{F}_p)][I_p\otimes\mathrm{F}_p\otimes I_p][\mathrm{F}_p\otimes I_p\otimes I_p],\end{aligned} \tag{16}$$

where $\mathrm{F}_p\otimes\mathrm{F}_p$ is the 2-D transform (14) and $(\mathrm{F}_p\otimes I_p)$ is p-D transform (15). The expression (16) is called the fast quantum DFT in the form of classical Fourier-Heisenberg-

Weyl transform. It represents a classical realization $p-1$ quantum transformations (for different Planck constants $\hbar=\alpha_3$).

Generalized quantum convolutions

Let $f[\theta]=\mathrm{QF}_x[\theta]\{\mathrm{Aw}[f\mid\theta]\}$, $\tilde{g}[\theta]=\mathrm{QF}_x[\theta]\{\mathrm{Aw}[g\mid\theta]\}$ and $F[\theta]=\mathrm{QF}_\omega[\theta]\{\mathrm{aW}[F\mid\theta]\}$, $G[\theta]=\mathrm{QF}_\omega[\theta]\{\mathrm{aW}[G\mid\theta]\}$ be two quantum signals and two quantum spectra. For the products $f[\theta]\cdot g[\theta]$ and $F[\theta]\cdot G[\theta]$ we have, respectively,

$$f[\theta]\cdot g[\theta]=\mathrm{QF}_x[\theta]\left\{\mathrm{Aw}[f\mid\theta]\underset{Cl}{\text{д}}\mathrm{Aw}[g\mid\theta]\right\},$$

$$F[\theta]\cdot G[\theta]=\mathrm{QF}_\omega[\theta]\left\{\mathrm{aW}[F(\omega\mid\theta)]\underset{Cl}{\#}\mathrm{aW}[G(\omega\mid\theta)]\right\},$$

where the expressions

$$\left(\mathrm{Aw}[f\mid\theta]\underset{Cl}{\text{д}}\mathrm{Aw}[g\mid\theta]\right)(\omega,x)=$$

$$=\int\limits_{(\nu,\tau)}\mathrm{Aw}[f\mid\theta]\left(\omega\underset{\theta}{\$}\nu,x'\underset{\theta}{}\tau\right)\mathrm{Aw}[g\mid\theta](\nu,\tau)\bar{\varphi}_\omega^{1/2}(\tau)\bar{\varphi}_\nu^{1/2}(x)\,d\mu(\nu)\,d\mu(\tau), \tag{17}$$

$$\left(\mathrm{aW}_\theta[F\mid\theta]\underset{Cl}{\#}\mathrm{aW}[G\mid\theta]\right)(x,\omega):=$$

$$=\int\limits_{(\nu,\tau)}\mathrm{aW}[F]\left(x'\underset{\theta}{}\tau,\omega\underset{\theta}{\$}\nu\right)\mathrm{aW}[G](\tau,\nu)\cdot\bar{\varphi}_\omega^{1/2}(\tau)\bar{\varphi}_\nu^{1/2}(x)\,d\mu(\nu)\,d\mu(\tau) \tag{18}$$

are called the *multiparameter generalized classical twisted signal* and *spectral convolutions*.

Conclusion

In this work we developed mathematical and algorithmically software interfaces between classical and quantum signal/image processing devices and systems, using for this goal a new generalized Weyl's quantization procedure in the form of generalized quantum Fourier transform. This interface can have classical and quantum realizations. Classical realization (on classical computer) of quantum Fourier transform gives classical realization of this interface.

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